1. Hodge Decomposition Theorem (Finite Dimensional Case)

A cochain complex of vector spaces (or simply a cochain complex) over a field k is a sequence of k-vector spaces $\{C^i : i \in \mathbb{Z}\}$ together with a sequence of k-linear maps $\{d^i : C^i \to C^{i+1} : i \in \mathbb{Z}\}$ such that $d^i \circ d^{i-1} = 0$ for any $i \in \mathbb{Z}$. A cochain complex over k is denoted by $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$. Since $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$, $\operatorname{Im} d^{i-1}$ is a vector subspace of $\ker d^i$. The i-th cohomology of a cochain complex $C^\bullet$ is defined to be the quotient vector space

$$H^i(C^\bullet) = \ker d^i / \operatorname{Im} d^{i-1}, \quad i \in \mathbb{Z}.$$

We denote $H^*(C^\bullet) = \bigoplus_{i\in\mathbb{Z}} H^i(C^\bullet)$ and $d = \bigoplus_{i\in\mathbb{Z}} d^i$.

In this note, all the vector spaces are assumed to be over the real field $k = \mathbb{R}$ or over the complex field $k = \mathbb{C}$.

Definition 1.1. A (Riemannian or Hermitian) metric h^C on a cochain complex C is a sequence of functions $\{h^i : C^i \times C^i \to k\}$ such that h^i is an inner product on C^i for each $i \in \mathbb{Z}$.

If $h^C = \{h^i\}$ is a metric on C, the inner product h^i is also denoted by $\langle\cdot,\cdot\rangle_{C^i}$ for each $i \in \mathbb{Z}$.

Let h^C be a metric on C. For each $i \in \mathbb{Z}$, we define a linear map $(d^{i+1})^* : C^{i+1} \to C^i$ by

$$\langle d^i x, y\rangle_{C^{i+1}} = \langle x, (d^{i+1})^* y\rangle_{C^i},$$

for any $x \in C^i$ and for any $y \in C^{i+1}$. We define a linear operator Δ^i on C^i by

$$\Delta^i = d^{i-1} \circ (d^i)^* + (d^{i+1})^* \circ d^i.$$

We call Δ^i the Laplace operator on C with respect to h^C.

Lemma 1.1. Let C be a cochain complex of vector spaces and h^C be a metric on C. Then

$$\ker \Delta^i = \ker d^i \cap \ker(d^i)^*.$$

Proof. If $x \in \ker d^i \cap \ker(d^i)^*$, then $d^i x = (d^i)^* x = 0$ which implies that $\Delta^i x = 0$. Therefore $x \in \ker \Delta^i$. This proves that

$$\ker \Delta^i \supseteq \ker d^i \cap \ker(d^i)^*.$$

Suppose $x \in \ker \Delta^i$. Then $\Delta^i x = 0$. Hence

$$\begin{aligned}\langle \Delta^i x, x\rangle_{C^i} &= \langle d^{i-1} \circ (d^i)^* x + (d^{i+1})^* \circ d^i x, x\rangle_{C^i} \\ &= \langle (d^i)^* x, (d^i)^* x\rangle_{C^i} + \langle d^i x, d^i x\rangle_{C^i} \\ &= \|(d^i)^* x\|^2_{C_i} + \|d^i x\|^2_{C_i} \\ &= 0.\end{aligned}$$

Since $\|(d^i)^* x\|^2_{C_i} \geq 0$ and $\|d^i x\|^2_{C_i} \geq 0$, we find that

$$\|(d^i)^* x\|^2_{C_i} = \|d^i x\|^2_{C_i} = 0$$

which implies that $(d^i)^* x = d^i x = 0$. Therefore $x \in \ker d^i \cap \ker(d^i)^*$. This proves that

$$\ker \Delta^i \subseteq \ker d^i \cap \ker(d^i)^*.$$

We complete the proof of our assertion. □

Theorem 1.1. (Hodge Decomposition Theorem) Let C be a cochain complex of finite dimensional vector spaces (i.e. each C^i is finite dimensional) and h^C be a metric on C. For each $i \in \mathbb{Z}$, we have the orthogonal decomposition of vector spaces

$$C^i = \ker \Delta^i \oplus \operatorname{Im} d^{i-1} \oplus \operatorname{Im}(d^{i+1})^*.$$

Proof. Let us prove the orthogonality of these subspaces of C^i.

Let $x \in \ker \Delta^i$ and $y \in \operatorname{Im} d^{i-1}$. Since $x \in \ker \Delta^i$, $d^i x = (d^i)^* x = 0$. Since $y \in \operatorname{Im} d^{i-1}$, we write $y = d^{i-1} z$ for $z \in C^{i-1}$. Therefore

$$\langle x, y\rangle_{C^i} = \langle x, d^{i-1}z\rangle_{C^i} = \langle (d^i)^* x, z\rangle_{C^{i-1}} = \langle 0, z\rangle_{C^{i-1}} = 0.$$

We prove that $\ker \Delta^i \perp \operatorname{Im} d^{i-1}$. Assume that $y' \in \operatorname{Im}(d^{i+1})^*$. We write $y' = (d^{i+1})^* z'$ for some $z' \in C^{i+1}$. Then

$$\langle x, y'\rangle_{C^i} = \langle x, (d^{i+1})^* z'\rangle_{C^i} = \langle d^i x, z'\rangle_{C^{i+1}} = \langle 0, z'\rangle_{C^{i+1}} = 0.$$

We prove that $\ker \Delta^i \perp \operatorname{Im}(d^{i+1})^*$. Let us compute $\langle y, y'\rangle_{C^i}$:

$$\langle y, y'\rangle_{C^i} = \langle d^{i-1}z, (d^{i+1})^* z'\rangle_{C^i} = \langle d^i \circ d^{i-1} z, z'\rangle_{C^{i+1}} = \langle 0, z'\rangle_{C^{i+1}} = 0$$

by $d^i \circ d^{i-1} = 0$. We prove that $\operatorname{Im} d^{i-1} \perp \operatorname{Im}(d^{i+1})^*$. We obtain that

$$\ker \Delta^i \oplus \operatorname{Im} d^{i-1} \oplus \operatorname{Im}(d^{i+1})^* \subseteq C^i.$$

Let us prove the inverse inclusion. It follows from the definition that $\Delta^i : C^i \to C^i$ is a self-adjoint operator. By spectral theorem, C^i can be decomposed into orthogonal direct sum of eigenspaces of Δ^i, i.e.

$$C^i = \bigoplus_{\lambda} E_{\Delta^i}(\lambda) = \ker \Delta^i \oplus \bigoplus_{\lambda \neq 0} E_{\Delta^i}(\lambda),$$

where $E_{\Delta^i}(\lambda) = \ker(\Delta^i - \lambda I)$. For each $x \in C^i$, we can write $x = x_0 + \sum_{\lambda \neq 0} x_\lambda$ (this is a finite sum because C^i is finite dimensional) where $x_0 \in \ker \Delta^i$ and $x_\lambda \in E_{\Delta^i}(\lambda)$ with $\lambda \neq 0$. Then $\Delta^i x = \sum_\lambda \lambda x_\lambda \in \bigoplus_{\lambda \neq 0} E_{\Delta^i}(\lambda)$. In other words, $\operatorname{Im} \Delta^i \subseteq \bigoplus_{\lambda \neq 0} E_{\Delta^i}(\lambda)$. For each $x = x_\lambda \in E_{\Delta^i}(\lambda)$, $\Delta^i x = \lambda x$ which implies that $x = \Delta^i(x/\lambda) \in \operatorname{Im} \Delta^i$. We find that $E_{\Delta^i}(\lambda) \subseteq \operatorname{Im} \Delta^i$ for any $\lambda \neq 0$. Since $\operatorname{Im} \Delta^i$ is a vector subspace of C^i, $\bigoplus_{\lambda \neq 0} E_{\Delta^i}(\lambda) \subseteq \operatorname{Im} \Delta^i$. We prove the equation

$$\operatorname{Im} \Delta^i = \bigoplus_{\lambda \neq 0} E_{\Delta^i}(\lambda).$$

We find that $C^i = \ker \Delta^i \oplus \operatorname{Im} \Delta^i$. For each $x \in C^i$, let us write $x = x_1 + x_2$ where $x_1 \in \ker \Delta^i$ and $x_2 \in \operatorname{Im} \Delta^i$. By definition, choose $y \in C^i$ so that

$$x_2 = \Delta^i y = d^{i-1}((d^i)^* y) + (d^{i+1})^*(d^i y) \in \operatorname{Im} d^{i-1} \oplus \operatorname{Im}(d^{i+1})^*.$$

This implies that $x \in \ker \Delta^i \oplus \operatorname{Im} d^{i-1} \oplus \operatorname{Im}(d^{i+1})^*$ for any $x \in C^i$. We find that

$$C^i \subseteq \ker \Delta^i \oplus \operatorname{Im} d^{i-1} \oplus \operatorname{Im}(d^{i+1})^*.$$

We complete the proof of our assertion. □

Corollary 1.1. Let C be a cochain complex of finite dimensional vector spaces and h^C be a metric on C. Then there is a linear isomorphism of vector spaces

$$H^i(C) \cong \ker \Delta^i \text{ for any } i \in \mathbb{Z}.$$

Proof. Since $\ker \Delta^i = \ker d^i \cap \ker(d^i)^*$, $\ker \Delta^i \subseteq \ker d^i$. Since $d^i \circ d^{i-1} = 0$, $\operatorname{Im} d^{i-1} \subseteq \ker d^i$. We find that $\ker \Delta^i \oplus \operatorname{Im} d^{i-1} \subseteq \ker d^i$. Let us prove that $\ker d^i \subseteq \ker \Delta^i \oplus \operatorname{Im} d^{i-1}$. Let $x \in \ker d^i$. By the Hodge decomposition, we can write $x = x_1 + x_2 + x_3$ for $x_1 \in \ker \Delta^i$ and for $x_2 \in \operatorname{Im} d^{i-1}$ and $x_3 \in \operatorname{Im}(d^{i+1})^*$. Choose $z \in C^{i+1}$ so that $x_3 = (d^{i+1})^* z$. Since $x \in \ker d^i$, $d^i x_3 = 0$. Now,

$$\|x_3\|^2_{C^i} = \langle x_3, x_3\rangle_{C^i} = \langle x_3, (d^{i+1})^* z\rangle_{C^i} = \langle d^i x_3, z\rangle_{C^{i+1}} = \langle 0, z\rangle_{C^{i+1}} = 0.$$

This proves that $x_3 = 0$. Therefore $x \in \ker \Delta^i \oplus \operatorname{Im} d^{i-1}$ for any $x \in \ker d^i$ and hence $\ker d^i \subseteq \ker \Delta^i \oplus \operatorname{Im} d^{i-1}$. We conclude that $\ker d^i = \ker \Delta^i \oplus \operatorname{Im} d^{i-1}$. By linear algebra, any (orthogonal) direct sum decomposition $U = V \oplus W$ of vector spaces induces a linear isomorphism $U/W \cong V$. We find that

$$H^i(C) = \ker d^i / \operatorname{Im} d^{i-1} \cong \ker \Delta^i.$$

□

The orthogonal decomposition $C^i = \ker \Delta^i \oplus \operatorname{Im} \Delta^i$ can also be proved using the following basic facts in linear algebra.

Lemma 1.2. Let $A : V \to W$ be a linear map between inner product spaces. Then

(1) $\operatorname{Im} A^\perp = \ker A^*$, and
(2) $(\operatorname{Im} A^*)^\perp = \ker A$.

Proof. Let $z \in \operatorname{Im} A^\perp$.Then $\langle y, z\rangle_W = 0$ for any $y \in \operatorname{Im} A$. For any $x \in V$,

$$\langle Ax, z\rangle_W = \langle x, A^* z\rangle_V = 0.$$

This implies that $A^* z = 0$. In other words, $z \in \ker A^*$. This proves that $\operatorname{Im} A^\perp \subseteq \ker A^*$. Suppose $z \in \ker A^*$. Then $A^* z = 0$. For any $y \in \operatorname{Im} A$, we can write $y = Ax$ and hence

$$\langle y, z\rangle_W = \langle Ax, z\rangle_W = \langle x, A^* z\rangle_V = \langle x, 0\rangle_V = 0.$$

We see that $z \in \operatorname{Im} A^\perp$. Therefore $\ker A^* \subseteq \operatorname{Im} A^\perp$. We conclude (1).

□

This lemma implies the following result.

Corollary 1.2. Let $A : V \to V$ be a self-adjoint linear operator on a finite dimensional inner product space. Then we have the following orthogonal decomposition:

$$V = \ker A \oplus \operatorname{Im} A.$$

Proof. Since A is self-adjoint, $A = A^*$. Hence $\operatorname{Im} A^\perp = \ker A$ which implies that $\ker A \cap \operatorname{Im} A = \{0\}$. Hence the sum of vector spaces $\ker A + \operatorname{Im} A$ is a direct sum $\ker A \oplus \operatorname{Im} A$. Since $\ker A \oplus \operatorname{Im} A$ is a vector subspace of V and by the rank-nullity lemma,

$$\dim V = \dim \ker A + \dim \operatorname{Im} A = \dim(\ker A \oplus \operatorname{Im} A),$$

we see that $V = \ker A \oplus \operatorname{Im} A$. (Here we use the fact that if W is a vector subspace of a finite dimensional vector space V so that $\dim V = \dim W$, then $W = V$.)

□