104 學年度 微積分競試

2016 年 4 月 23 日

(Solution)

姓名：＿＿＿＿＿＿ 學號：＿＿＿＿＿＿ 學系：＿＿＿＿＿＿

說明:

1. 本試題含封面共 9 頁，8 大題。
2. 考試時間 **100** 分鐘。
3. 請在每個試題所屬的頁面作答。如欲使用試題背面，請標示清楚。
4. 清楚地寫出計算及證明的過程，沒有過程的答案將不予記分。

題號	配分	分數
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	20	
8	10	
總分	100	

1. 計算下列積分

(a) (5 points)

$$\int x \ln x \, dx =$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C.$$

(b) (5 points)

$$\int_1^2 x(2x-1)^{-6} dx =$$

Let $u = 2x - 1.$ $\Rightarrow$ $dx = \frac{1}{2} du,$ $x = \frac{1}{2}(u+1).$

$$\int_1^2 x(2x-1)^{-6} dx = \int_1^3 \frac{1}{4}(u+1) u^{-6} \, du$$

$$= \frac{1}{4}\left[-\frac{1}{4} u^{-4} - \frac{1}{5} u^{-5} \right]\Big|_1^3$$

$$= \frac{1}{4}\left[-\frac{1}{4}\frac{1}{3^4} - \frac{1}{5}\frac{1}{3^5} + \frac{1}{4} + \frac{1}{5} \right].$$

2. (10 points) $f(x)$ 是連續函數，且 $\forall x \in [0,1]$, $0 < f(x) < 1$。請證明至少存在一個數 $c \in [0,1]$ 使得 $f(c) = c$。

Let $F(x) = f(x) - x$.

Then $F(0) = f(0) > 0$.

$F(1) = f(1) - 1 < 0$.

Hence, by the intermediate value theorem,

$\exists$ at least one $c \in (0,1)$ s.t.

$F(c) = 0$.

i.e. $f(c) = c$.

3. 函數 f 定義為

$$f(x) = \sqrt{|x-1|}, \quad x \in R.$$

(a) (10 points) 請討論 $f(x)$ 的可微性並求出一次導數。

Case1. $x>1$. $f(x)=\sqrt{x-1}$, $f'(x)=\frac{1}{2}(x-1)^{-\frac{1}{2}}$.

Case2. $x<1$. $f(x)=\sqrt{1-x}$, $f'(x)=\frac{-1}{2}(1-x)^{-\frac{1}{2}}$.

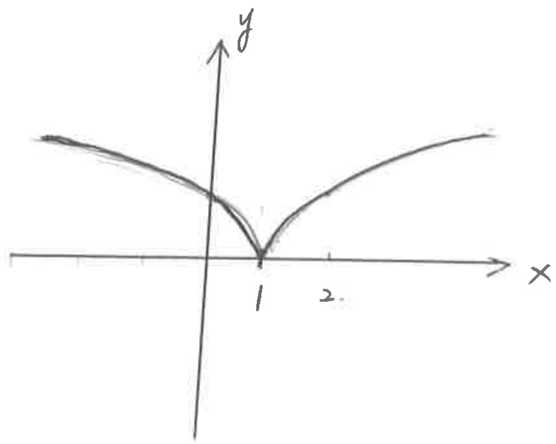
power function

Case3. $x=1$. Check that $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$ doesn't exist. So, $f'(1)$ doesn't exist.

$f(x)$ is not differentiable at $x=1$.

(b) (10 points) 請描繪 $f(x)$ 的圖形。

	$x>1$	$x<1$
f'	+	−
f''	−	−



For $x>1$, $f''(x) = -\frac{1}{4}(x-1)^{-\frac{3}{2}}$

For $x<1$, $f''(x) = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

4. 計算下列極限。

(a) (5 points) $\lim_{x\to 1} x^{1/(x-1)}$

Let $f(x) = x^{\frac{1}{x-1}}$.

$\ln f(x) = \frac{1}{x-1} \ln x$.

$\Rightarrow \lim_{x\to 1} \ln f(x) = \lim_{x\to 1} \frac{\ln x}{x-1}$

$= \lim_{x\to 1} \frac{1}{x} = 1$.

$\Rightarrow \lim_{x\to 1} f(x) = \lim_{x\to 1} e^{\ln f(x)} = e$.

(b) (5 points) $\lim_{x\to 0} \frac{\sqrt{2x+4}-2}{x} = \lim_{x\to 0} \frac{2x+4-4}{x(\sqrt{2x+4}+2)}$

$= \lim_{x\to 0} \frac{2}{\sqrt{2x+4}+2}$

$= \frac{1}{2}$.

5. (10 points) $p > 1$ 是一個正實數。請證明

$$(1+x)^p > 1 + px, \quad \forall x \in (0, \infty).$$

Let $f(x) = (1+x)^p - 1 - px$.

$f(0) = 0$, $\quad f'(x) = p(1+x)^{p-1} - p$.

$f'(0) = 0$.

And $f'(x) > 0$ for $x > 0$.

For any given $x > 0$, by the mean value theorem, there exists some point $\xi \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x} = f'(\xi).$$

Hence $\quad f(x) - f(0) > 0$.

i.e. $\quad f(x) > 0$.

$\Rightarrow \quad (1+x)^p > 1 + px, \quad \forall x \in (0, \infty)$.

6. (10 points) 請推導出半徑為 r 的球體體積。

Let $f(x) = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$.

Volume of sphere $= \int_{-r}^{r} \pi (f(x))^2 \, dx$.

$= \int_{-r}^{r} \pi (r^2 - x^2) \, dx$.

$= \frac{4}{3} \pi r^3$.

7. 判斷下列瑕積分是否收斂。

(a) (10 points) $\displaystyle\int_{-3}^{3} \frac{1}{x(x+1)}dx$

$$\int_{-3}^{3} \frac{1}{x(x+1)}dx = \left(\int_{-3}^{-1} + \int_{-1}^{0} + \int_{0}^{3}\right) \frac{1}{x(x+1)}dx.$$

We can check that $\displaystyle\int_{0}^{3} \frac{1}{x(x+1)}dx$ is divergent.

$$\left(\because \int_{0}^{3} \frac{1}{x(x+1)}dx > \int_{0}^{3} \frac{1}{4x}dx, \text{ and } \int_{0}^{3} \frac{1}{x}dx \text{ is divergent.} \right)$$

Hence, $\displaystyle\int_{-3}^{3} \frac{1}{x(x+1)}dx$ is divergent.

(b) (10 points) $\displaystyle\int_{0}^{\pi/2} \ln(\tan x)dx$ $= 0$.

Let $y = \tan x$, $dy = \sec^2 x\, dx \Rightarrow dx = \frac{1}{1+y^2}dy$.

$$\int_{0}^{\pi/2} \ln(\tan x)dx = \int_{0}^{1} \frac{\ln y}{1+y^2}dy + \int_{1}^{\infty} \frac{\ln y}{1+y^2}dy.$$

(1) $\displaystyle\int_{0}^{1} \frac{\ln y}{1+y^2}dy = -\int_{1}^{\infty} \frac{\ln y}{1+y^2}dy.$ (Let $t = \frac{1}{y}$.)

(2) There exists some constant $M > 1$ s.t.

$$\ln y < \sqrt{y}, \text{ for } y > M.$$

$$\int_{1}^{\infty} \frac{\ln y}{1+y^2}dy = \int_{1}^{M} \frac{\ln y}{1+y^2}dy + \int_{M}^{\infty} \frac{\ln y}{1+y^2}dy.$$

And $\displaystyle\int_{M}^{\infty} \frac{\ln y}{1+y^2}dy < \int_{M}^{\infty} y^{-\frac{3}{2}}dy < \infty.$

So, $\displaystyle\int_{1}^{\infty} \frac{\ln y}{1+y^2}dy$ is convergent.

8. (10 points) $f(x) = e^x$, $x \in R$. 求出 $f(x)$ 在 $x = 0$ 的 n 階 Taylor 展式。

$$f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x.$$

The Taylor polynomial of degree n, $P_n(x)$, is written as

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

$$= 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n.$$