九十年

微積分競試參考答案

一、(甲) 多重選擇題 (共 5 小題，每小題 4 分，每小題可能有一個或一個以上的答案，全對才給分). (20 分)

(1) (a), (b), (e).

(2) (e).

(3) (a), (b), (d).

(4) (b).

(5) (a), (b), (e).

一、(乙) 填充題 (共 5 小題，每小題 4 分). (20 分)

(1) $\frac{1}{n!}$.

(2) $\frac{1}{5}$.

(3) $\frac{1}{2}$.

(4) $\frac{\pi}{4}$.

(5) $y = -\frac{3}{2}(x-1)$.

二、 (1)

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+2x}}\,dx &= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x}}\,dx - \int \frac{1}{\sqrt{x^2+2x}}\,dx \\ &= \sqrt{x^2+2x} - \int \frac{1}{\sqrt{x^2+2x}}\,dx.\end{aligned}$$

對於上式第二項之不定積分，設 $f(x) = \frac{1}{\sqrt{x^2+2x}}$，顯然其定義域為 $(-\infty,-2)\cup(0,+\infty)$，

- 若被積函數之定義域 $(0,+\infty)$，則

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+2x}}\,dx &= \int \frac{1}{\sqrt{(x+1)^2-1}}\,dx \\
&= \int \frac{1}{\sqrt{u^2-1}}\,du\,, \quad (\text{令 } u = x+1) \\
&= \int \frac{\sec t \tan t\,dt}{\sqrt{\sec^2 t - 1}}\,, \quad [\text{令 } u = \sec t,\ t \in (0, \pi/2)] \\
&= \int \frac{\sec t \tan t\,dt}{\tan t} = \int \sec t\,dt \\
&= \ln|\sec t + \tan t| + C \\
&= \ln|u + \sqrt{u^2-1}| + C \\
&= \ln|x + 1 + \sqrt{x^2+2x}| + C
\end{aligned}
$$

- 若被積函數之定義域 $(-\infty, -2)$，則

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+2x}}\,dx &= \int \frac{1}{\sqrt{(x+1)^2-1}}\,dx \\
&= \int \frac{1}{\sqrt{u^2-1}}\,du\,, \quad (\text{令 } u = x+1) \\
&= \int \frac{\sec t \tan t\,dt}{\sqrt{\sec^2 t - 1}}\,, \quad [\text{令 } u = \sec t,\ t \in (\pi/2, \pi)] \\
&= \int \frac{\sec t \tan t\,dt}{-\tan t} = -\int \sec t\,dt \\
&= -\ln|\sec t + \tan t| + C \\
&= -\ln|u - \sqrt{u^2-1}| + C \\
&= -\ln|x + 1 - \sqrt{x^2+2x}| + C \\
&= \ln\left|\frac{1}{x+1-\sqrt{x^2+2x}}\right| + C \\
&= \ln|x + 1 + \sqrt{x^2+2x}| + C'.
\end{aligned}
$$

(2) 令 $I = \displaystyle\int_0^1 \frac{\ln(1+x)}{\ln(1+x)+\ln(2-x)}\,dx$，其次，令 $u = 1-x$，則 $x = 1-u$，

$$
I = -\int_1^0 \frac{\ln(2-u)}{\ln(2-x)+\ln(1+u)}\,du = \int_0^1 \frac{\ln(2-u)}{\ln(2-u)+\ln(1+u)}\,du.
$$

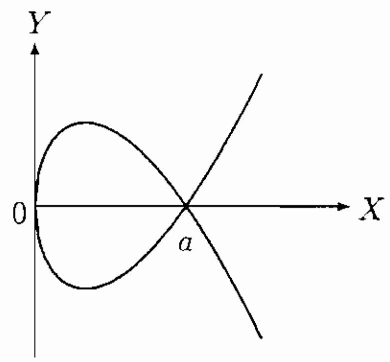
是以

$$
\begin{aligned}
2I = I + I &= \int_0^1 \frac{\ln(1+x)}{\ln(1+x)+\ln(2-x)}\,dx + \int_0^1 \frac{\ln(2-u)}{\ln(2-u)+\ln(1+u)}\,du \\
&= \int_0^1 \frac{\ln(1+x)+\ln(2-x)}{\ln(2-x)+\ln(1+x)}\,dx = \int_0^1 dx = 1.
\end{aligned}
$$

得 $I = 1/2$.

三 (1) 由於

$$y^2 = x(x-a)^2 \;\Leftrightarrow\; y = \pm\sqrt{x}(x-a),$$



其所圍部分為

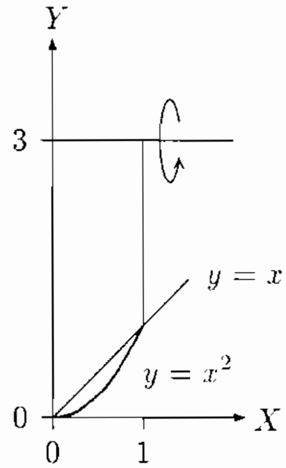
$$R = \{(x,y) \mid 0 \le x \le a,\ \sqrt{x}(x-a) \le y \le -\sqrt{x}(x-a)\},$$

故所求之面積為

$$
\begin{aligned}
A &= \int_0^a (-\sqrt{x}(x-a) - \sqrt{x}(x-a))dx \\
&= 2\int_0^a \sqrt{x}(a-x)\,dx = 2\int_0^a (ax^{1/2} - x^{3/2})\,dx \\
&= 2\Big[\frac{ax^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\Big]_0^a = \frac{8}{15}a^{5/2}.
\end{aligned}
$$

(2) 先求交點： $x^2 = x \;\Leftrightarrow\; x \in \{0,1\}$. 是以所求之旋轉體的體積為

$$
\begin{aligned}
V &= \pi\int_0^1 [(3-x^2)^2 - (3-x)^2]dx = \pi\int_0^1 (x^4 - 7x^2 + 6x)dx \\
&= \pi\Big[\frac{x^5}{5} - \frac{7x^3}{3} + \frac{6x^2}{2}\Big]_0^1 \\
&= \frac{13\pi}{15}
\end{aligned}
$$



(3) 上述旋轉體之表面面積為

$$A = 2\pi \int_0^1 (3-x)\sqrt{1+1^2}\,dx + 2\pi \int_0^1 (3-x^2)\sqrt{1+(2x)^2}\,dx.$$

四、 $f(x) = (x-1)^{1/3} - 2(x-1)^{4/3}$ 的定義域顯然為 $\mathbb{R}$，其次，

$$f'(x) = \frac{1}{3}(x-1)^{-2/3} - \frac{8}{3}(x-1)^{1/3} = \frac{9-8x}{3(x-1)^{2/3}}, \quad \forall x \neq 1.$$

$$f''(x) = \frac{-2}{9}(x-1)^{-5/3} - \frac{8}{9}(x-1)^{-2/3} = \frac{6-8x}{9(x-1)^{5/3}}, \quad \forall x \neq 1,$$

顯然

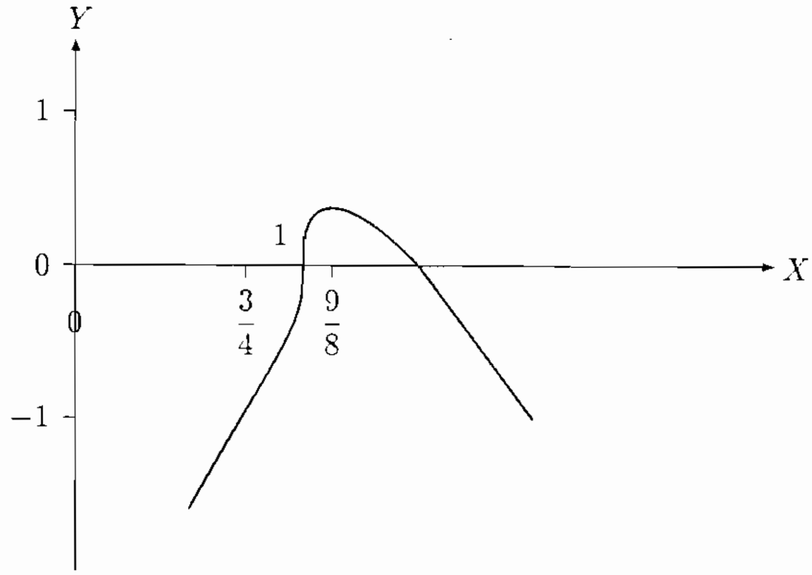
$$f'(x) = 0 \Leftrightarrow x = \frac{9}{8},$$

$$f''(x) = 0 \Leftrightarrow x = \frac{3}{4}.$$

且 f 在點 1 不為可微. 列表如下：

x		3/4		1		9/8	2/3	
$f(x)$		−		0		+	0	+
$f'(x)$		+			+	0	−	
$f''(x)$	−	0	+			−		
說 明	⌢	↗ 反曲	⌣	反曲	↗	max ⌢	↘	

其圖形為



五、 (1) (a) 由於

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0}(x^2e^x)^{x/(1+x)}\\ &= \lim_{x\to 0}\exp\left(\frac{x}{1+x}\ln(x^2e^x)\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\ln(x^2e^x)}{1+\frac{1}{x}}\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\frac{1}{x^2e^x}(2xe^x+x^2e^x)}{-\frac{1}{x^2}}\right), \quad \text{(l'Hospital)}\\ &= \exp\left(\lim_{x\to 0} -(2x+x^2)\right) = \exp 0 = 1 = f(0).\end{aligned}$$

知 f 在 $x=0$ 為連續.

(b) 由於

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{(x^2e^x)^{x/(1+x)}-1}{x}\\ &= \lim_{x\to 0^+}\frac{\exp\left(\frac{x}{1+x}\ln(x^2e^x)\right)-1}{x}, \quad \text{(由 (a) 知其為不定型)}\\ &= \lim_{x\to 0^+}\exp\left(\frac{x}{1+x}\ln(x^2e^x)\right)\frac{(1+x)(2x+2\ln x+2)-x(2\ln x+x)}{(1+x)^2}\\ &\qquad\qquad \text{(利用 l'Hospital)}\\ &= \lim_{x\to 0^+}\exp\left(\frac{x}{1+x}\ln(x^2e^x)\right)\frac{2(1+x)^2+2\ln x-2x^2}{(1+x)^2}\\ &= -\infty.\end{aligned}$$

知 f 在 $x=0$ 不為可微.

(2) 令 $u=f^{-1}(x)$, $dv=dx$, 則 $du=d\,f^{-1}(x)$, $v=x$, 代入則

$$\begin{aligned}\int f^{-1}(x)\,dx &= xf^{-1}(x)-\int x\,d\,f^{-1}(x)\\ &= xf^{-1}(x)-\int f\Big(f^{-1}(x)\Big)\,d\,f^{-1}(x)\\ &= xf^{-1}(x)-F\Big(f^{-1}(x)\Big)+C.\end{aligned}$$

(3) 法一: 利用 (2)

$$\int \operatorname{sech}^{-1}x\,dx = x\operatorname{sech}^{-1}x-F(\operatorname{sech}^{-1}x)+C.$$

其中

$$\begin{aligned}F(x)=\int \operatorname{sech}x\,dx &= \int\frac{\cosh x\,dx}{\cosh^2 x}\\ &= \int\frac{d(\sinh x)}{1+\sinh^2 x}=\tan^{-1}(\sinh x)+C.\end{aligned}$$

法二: 先微分 $\operatorname{sech}^{-1}x$,

$$\begin{aligned}\frac{d}{dx}\operatorname{sech}^{-1}x &= \frac{1}{\frac{d}{dy}\operatorname{sech}y},\quad (x=\operatorname{sech}y)\\ &= \frac{1}{-\tanh y\cdot\operatorname{sech}y}\\ &= -\frac{1}{x\sqrt{1-x^2}}.\end{aligned}$$

其次利用 by parts, 令 $u=\operatorname{sech}^{-1}x$, $dv=dx$, 則

$$du=-\frac{dx}{x\sqrt{1-x^2}},\ v=x,$$

$$\begin{aligned}\int \operatorname{sech}^{-1}x\,dx &= x\operatorname{sech}^{-1}x+\int\frac{x}{x\sqrt{1-x^2}}\,dx\\ &= x\operatorname{sech}^{-1}x+\int\frac{1}{\sqrt{1-x^2}}\,dx\\ &= x\operatorname{sech}^{-1}x+\sin^{-1}x+C.\end{aligned}$$