95學年度基礎學科微積分競試試題解答

1. 解：(1) 因 $\sum_{n=1}^{\infty} a_n$ 收斂，故 $\lim_{n\to\infty} a_n = 0$，則存在 N 使得 $0 < a_n < 1$，當 $n \geq N$，

又因 $\sum_{n=N}^{\infty} a_n^2 \leq \sum_{n=N}^{\infty} a_n$ 收斂，所以 $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2$ 收斂。

(2) $\sum_{n=1}^{\infty} \sqrt{a_n}$ 可能收斂，也可能發散。

例子：$\sum_{n=1}^{\infty} \frac{1}{n^2}$ 收斂，且 $\sum_{n=1}^{\infty} \frac{1}{n}$ 發散；$\sum_{n=1}^{\infty} \frac{1}{n^4}$ 收斂和 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 收斂。 ///

2. 解：（方法一）

因 $f(x)$ 爲嚴格單調可微函數，故反函數 f^{-1} 存在，且爲爲嚴格單調可微函數，同時 $(f^{-1}(y))'f'(x) = 1$，利用分部積分

$$\int_{f(b)}^{f(a)} f^{-1}(y)\,dy = yf^{-1}(y)\Big|_{f(a)}^{f(b)} - \int_{f(a)}^{f(b)} y(f^{-1}(y))'\,dy$$

利用變數變換 $y = f(x)$，則

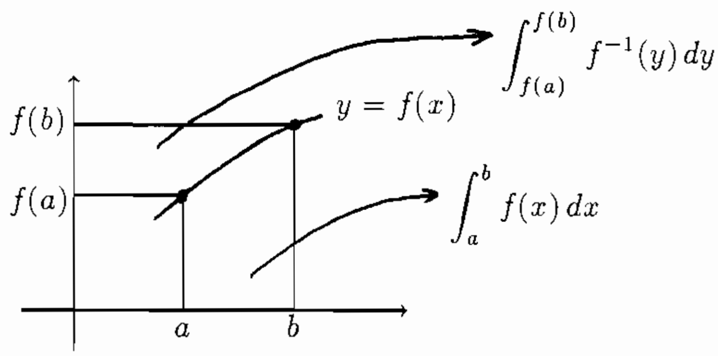
$$\int_{f(a)}^{f(b)} (f^{-1}(y))'\,dy = \int_a^b f(x)\frac{1}{f'(x)}f'(x)\,dx = \int_a^b f(x)\,dx = F(b)$$

故可得

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a) - F(b)$$

（方法二）

若 f 遞增，



$$\int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) - \int_a^b f(x)\,dx$$
$$= bf(b) - af(a) - F(b)$$

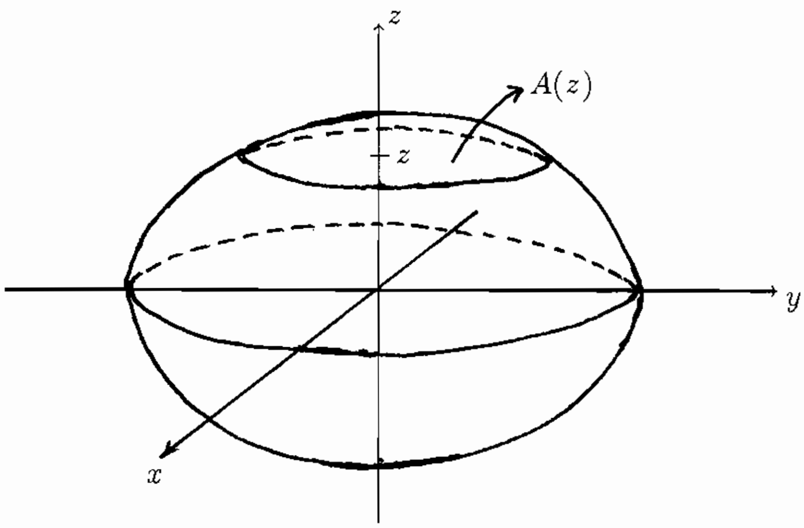
若 f 遞減，證明類似。

3. 解：Torus的體積爲 $V = \pi a^2 2\pi b^2$，因 $2x : 2a = \theta : \frac{\pi}{2}$, 故 $x = \frac{2a}{\pi}\theta$

又 $2y : 2b = \theta : \frac{\pi}{2}$, 故 $y = \frac{2b}{\pi}\theta$，橢圓面積 $A(\theta) = \pi xy = \frac{4ab}{\pi}\theta^2$

$$
\begin{aligned}
\text{螺殼體積} V &= \int_0^{\frac{\pi}{2}} A(\theta)(r + x)\, d\theta \\
&= \int_a^b r\frac{4ab}{\pi}\theta^2 + \frac{8a^2b}{\pi^2}\theta^3\, d\theta \\
&= \frac{\pi^2 abr}{6} + \frac{\pi^2 a^2 b}{8}
\end{aligned}
$$

4. 解：



因橢圓體 $\frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} + \frac{(z-c)^2}{C^2} = 1$ 平移 (a, b, c)

可得橢圓體 $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$, 故體積相同。

固定 z 設橢圓截面積 $= A(z)$, 因 $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 - \frac{z^2}{C^2}$,

則 $\frac{x^2}{A^2(1 - \frac{z^2}{C^2})} + \frac{y^2}{B^2(1 - \frac{z^2}{C^2})} = 1$，故 $A(z) = \pi A\sqrt{1 - \frac{z^2}{C^2}}B\sqrt{1 - \frac{z^2}{C^2}}$

橢圓體的體積：

$$V = \int_{-C}^{C} A(z)\, dz = \int_{-C}^{C} \pi AB(1 - \frac{z^2}{C^2})\, dz = \frac{4}{3}\pi ABC$$

5. 解：

由微積分基本定理

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

用分部積分,

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt$$

再次用分部積分

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{1}{2!}\int_0^x f'''(t)(x-t)^2\,dt$$

6. 解：

(a) $$\frac{dy}{dx} = \frac{3\sinh(x^2+4)(2x)\sqrt{x^2+3x+1} - 3\cosh(x^2+4)\frac{1}{2}\frac{2x+3}{\sqrt{x^2+3x+1}}}{x^2+3x+1}$$

(b) $y = \sinh x = \dfrac{e^x - e^{-x}}{2} \Rightarrow e^x = y + \sqrt{y^2+1}$和$e^{-x} = -y + \sqrt{y^2+1}$

又$\sinh^{-1}\sinh x = x$,則$(\sinh^{-1} y)' = \dfrac{1}{\cosh x} = \dfrac{1}{\sqrt{y^2+1}}$

對於$y = \sinh^{-1} e^x$,用連鎖法則,

$$\begin{aligned}\frac{dy}{dx} &= (\sinh^{-1} y)'\Big|_{y=e^x} \cdot e^x \\ &= \frac{1}{\sqrt{y^2+1}}\Big|_{y=e^x} \cdot e^x = \frac{e^x}{\sqrt{e^{2x}+1}}\end{aligned}$$

(c) 令$F(x) = \displaystyle\int_0^x \sin(t^3)e^{-t^3}\,dt$,根據微積分基本定理,

$$F'(x) = \sin(x^3)e^{-x^3},$$

且

$$y = \int_{x^2}^{x^3} \sin(t^3)e^{-t^3}\,dt = F(x^3) - F(x^2)$$

現在求導數,由連鎖法則

$$\begin{aligned}\frac{dy}{dx} &= F'(x^3)(3x^2) - F'(x^2)(2x) \\ &= 3x^2\sin(x^9)e^{-x^9} - 2x\sin(x^6)e^{-x^6}\end{aligned}$$

7. **解：**

(a)
$$\int x^2 \tan^{-1} x^3\, dx \qquad \text{令 } u = x^3 \text{ 且 } du = 3x^2\, dx$$
$$= \frac{1}{3}\int \tan^{-1} u\, du \qquad \text{用分部積分}$$
$$= \frac{1}{3}\left[u\tan^{-1} u - \int \frac{u}{1+u^2}\, du\right]$$
$$= \frac{1}{3}x^3\tan^{-1} x^3 - \frac{1}{6}\ln(1+x^6) + C \qquad ///$$

(b)
$$\int \frac{u}{u^6-8}\, du \qquad (x = u^2)$$
$$= \frac{1}{2}\int \frac{1}{x^3-8}\, dx$$
$$= \frac{1}{24}\int \frac{1}{x-2} - \frac{1}{24}\int \frac{x+4}{x^2+2x+4}\, dx$$
$$= \frac{1}{24}\ln|x-2| - \frac{1}{24}\int \frac{x+1}{x^2+2x+4}\, dx - \frac{1}{8}\int \frac{1}{(x+1)^2+3}\, dx \qquad (x+1 = \sqrt{3}V)$$
$$= \frac{1}{24}\ln|x-2| - \frac{1}{48}\ln|x^2+2x+4| - \frac{1}{8}\int \frac{1}{(\sqrt{3}V)^2+3}\sqrt{3}\, dV$$
$$= \frac{1}{24}\ln|x-2| - \frac{1}{48}\ln|x^2+2x+4| - \frac{1}{8\sqrt{3}}\tan^{-1} V + C$$
$$= \frac{1}{24}\ln|u^2-2| - \frac{1}{48}\ln|u^4+2u^2+4| - \frac{1}{8\sqrt{3}}\tan^{-1}\frac{u^2+1}{\sqrt{3}} + C$$

(c)
$$\int \sin^6 x\, dx \qquad \cos 2x = \cos^2 x - \sin^2 x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 x - \cos^3 2x\, dx$$
$$= \frac{1}{8}\left[x - \frac{3}{2}\sin 2x + \frac{3}{2}\int 1+\cos 4x\, dx - \int \cos 2x - \sin^2 2x\cos 2x\, dx\right]$$
$$= \frac{1}{8}\left[x - \frac{3}{2}\sin 2x + \frac{3}{2}x + \frac{3}{8}\sin 4x - \frac{1}{2}\sin 2x + \frac{1}{6}\sin^3 2x\right] + C$$
$$= \frac{1}{8}\left[\frac{5}{2}x - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right] + C$$

8. 解：

(a) 因 $\dfrac{dx}{dt}=\dfrac{-4t}{(1+t^2)^2}\begin{cases}>0 & \text{當 } -\infty<t<0,\ x\text{向右移}\\ =0 & \text{當 } t=0,\\ <0 & \text{當 } 0<t<\infty,\ x\text{向左移}\end{cases}$

所以當 $-\infty<t\le 0$ 時, y 爲 x 的函數,

或當 $0\le t<\infty$ 時, y 爲 x 的函數。

當 $-\infty<t\le 0$ 時, $-1<x(t)\le +1$,所以定義域 $=(-1,1]$ 。

(b) (方法一)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{t^4+4t^2-1}{4t}=0$$

$$\Rightarrow \frac{dy}{dt}=\frac{(t^2+2+\sqrt{5})(t+\sqrt{\sqrt{5}-2})(t-\sqrt{\sqrt{5}-2})}{4t}\begin{cases}<0 & \text{當 } -\infty<t<-\sqrt{\sqrt{5}-2}\\ =0 & \text{當 } t=-\sqrt{\sqrt{5}-2}\\ >0 & \text{當 } -\sqrt{\sqrt{5}-2}<t<0\end{cases}$$

極值點 $\left(x(-\sqrt{\sqrt{5}-2}),y(-\sqrt{\sqrt{5}-2})\right)=\left(\frac{\sqrt{5}-1}{2},-\sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}\right)$

$$y=f(x)\begin{cases}\text{遞減 當 } & t<-\sqrt{\sqrt{5}-2}\\ \text{遞增 當 } & -\sqrt{\sqrt{5}-2}<t<0\end{cases}$$

$$t\to-\infty\Rightarrow x(t)=\frac{1-t^2}{1+t^2}\to-1$$

$$y(t)=t\frac{1-t^2}{1+t^2}\to+\infty$$

有垂直漸近線 $x=-1$ 。

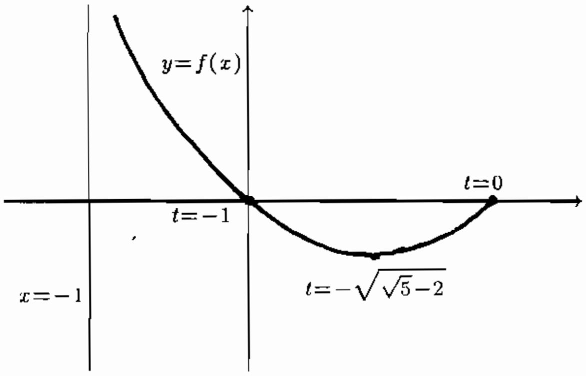
$$x(0)=1 \qquad y(0)=0$$

$$x(-1)=0 \qquad y(-1)=0$$

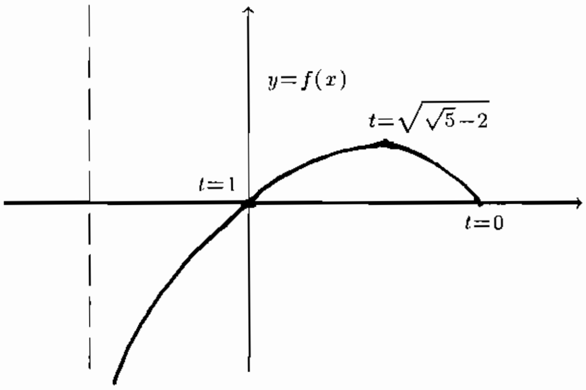
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{\frac{d}{dt}\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}{\frac{dx}{dt}}$$

$$=\frac{\frac{d}{dt}\frac{t^4+4t^2-1}{4t}}{\frac{-4t}{(1+t^2)^2}}=\frac{-(1+t^2)^2}{16}\frac{3t^4+4t^2-1}{t^3}\le 0$$

當 $t < 0$, 則 $y = f(x)$ 上凹 $x \in (-1, 1]$



當 $0 \le t < \infty$,



(方法二)

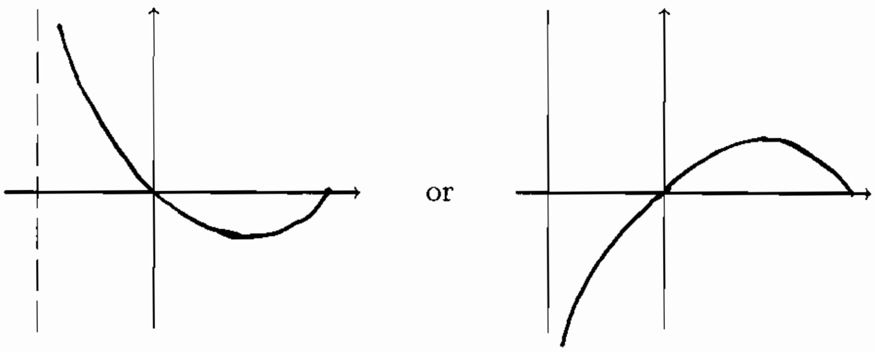
因 $x(t) = \dfrac{1-t^2}{1+t^2}$ 和 $y(t) = t\dfrac{1-t^2}{1+t^2}$ $\Rightarrow y = \pm x\sqrt{\dfrac{1-x}{1+x}}$ $\quad -1 < x \le 1$

計算 $\dfrac{dy}{dx} = \pm\dfrac{1-x-x^2}{(1+x^2)}\sqrt{\dfrac{1+x}{1-x}}$

$$\frac{d^2y}{dx^2} = \pm\frac{2x^2+3x-4}{(1+x)^2(1-x)^2}\sqrt{\frac{1-x}{1+x}}$$

垂直漸近線：$x = -1$

作圖：



9.

解：因為

$$\begin{aligned} a_n &= \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} + \cdots + \frac{1}{\sqrt{nn}} \\ &= \Big(\frac{1}{\sqrt{\frac{1}{n}}} + \frac{1}{\sqrt{\frac{2}{n}}} + \cdots + \frac{1}{\sqrt{\frac{n}{n}}}\Big)\frac{1}{n} \end{aligned}$$

此為瑕積分 $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ 的黎曼和,則

$$\begin{aligned} \lim_{n\to\infty} a_n &= \int_0^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{a\to 0^+} 2\sqrt{1} - 2\sqrt{a} = 2 \end{aligned}$$

10. 解：對於 $x \neq 0, x^3, \sin x, \frac{1}{x^2}$ 皆為可微分函數,故 $x^3 \sin \frac{1}{x^2}$ 為可微分函數且

$$f'(x) = 3x^2 \sin \frac{1}{x^2} - 2\cos \frac{1}{x^2}$$

對於 $x = 0$,因極限存在

$$\frac{f(x) - f(0)}{x - 0} = x^2 \sin \frac{1}{x^2} \to 0 \text{ 當 } x \to 0$$

故 $f'(0) = 0$。

但是 $\lim_{x\to 0} f'(x)$ 不存在,所以 $f'(x)$ 除0點外為連續函數。 ///