Chapter 8: Techniques of Integration

Section 8.1 Integral Tables and Review
a. Important Integrals
b. Example
c. Integral Tables

Section 8.2 Integration by Parts
a. Formulas for Integration by Parts
b. Example
c. The Definite Integral
d. Log and Inverse Trigonometric Functions
e. Example

Section 8.3 Powers and Products of Trigonometric Functions
a. Basic Trigonometric Identities
b. Powers of Sines and Cosines
c. Example
d. Powers of Tangent
e. Powers of Secant
f. Powers of Cotangent and Cosecant

Section 8.4 Integrals Involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$
a. Formulas for Integration
b. Trigonometric Substitutions
c. Example
d. Formula for Quadratics under a Radical Sign
e. Example
f. Reduction Formula

Section 8.5 Rational Functions; Partial Fractions
a. Partial Fractions
b. Distinct Linear Factors
c. Repeated Linear Factors
d. Irreducible Quadratic Factors
e. Repeated Irreducible Quadratic Factors

Section 8.6 Some Rationalizing Substitutions
a. Rationalizing Substitutions
b. Example

Section 8.7 Numerical Integration
a. Estimating Definite Integrals
b. Left-endpoint Estimate
c. Right-endpoint Estimate
d. Midpoint Estimate
e. Trapezoidal Estimate
f. Parabolic Estimate (Simpson's Rule)
g. Example
h. Error Estimates
i. Theoretical Error
j. Theoretical Error for the Trapezoidal Rule
k. Theoretical Error for Simpson's Rule
l. Example

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integral Tables and Review

1. $\displaystyle\int k\,du = ku + C, \qquad k \text{ constant.}$

2. $\displaystyle\int u^r\,du = \frac{u^{r+1}}{r+1} + C, \qquad r \text{ constant}, r \neq -1.$

3. $\displaystyle\int \frac{1}{u}du = \ln|u| + C.$

4. $\displaystyle\int e^u\,du = e^u + C.$

5. $\displaystyle\int p^u\,du = \frac{p^u}{\ln p} + C, \qquad p > 0 \text{ constant}, p \neq 1.$

6. $\displaystyle\int \sin u\,du = -\cos u + C.$

7. $\displaystyle\int \cos u\,du = \sin u + C.$

8. $\displaystyle\int \tan u\,du = \ln|\sec u| + C.$

9. $\displaystyle\int \cot u\,du = \ln|\sin u| + C.$

10. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C.$

11. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C.$

12. $\displaystyle\int \sec u \tan u\,du = \sec u + C.$

13. $\displaystyle\int \csc u \cot u\,du = -\csc u + C.$

14. $\displaystyle\int \sec^2 u\,du = \tan u + C.$

15. $\displaystyle\int \csc^2 u\,du = -\cot u + C.$

16. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}\,du = \arcsin\frac{u}{a} + C, \qquad a > 0 \text{ constant.}$

17. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C, \qquad a > 0 \text{ constant.}$

18. $\displaystyle\int \frac{du}{|u|\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C, \qquad a > 0 \text{ constant.}$

19. $\displaystyle\int \sinh u\,du = \cosh u + C.$

20. $\displaystyle\int \cosh u\,du = \sinh u + C.$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integral Tables and Review

Example

Calculate $\displaystyle\int_0^1 \frac{e^x}{e^x+2}\,dx$

Solution

Set $u = e^x + 2$, $du = e^x dx$. At $x = 0$, $u = 3$; at $x = 1$, $u = e + 2$.

Thus

$$\int_0^1 \frac{e^x}{e^x+2}\,dx = \int_3^{e+2} \frac{du}{u} = \Big[\ln|u|\Big]_3^{e+2}$$

Formula 3

$$= \ln(e+2) - \ln 3 = \ln\left[\tfrac{1}{3}(e+2)\right] \cong 0.45.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integral Tables and Review

Using a Table of Integrals A table of over 100 integrals, including those listed at the beginning of this section, appears on the inside covers of this text. This is a relatively short list. Mathematical handbooks such as *Burington's Handbook of Mathematical Tables and Formulas* and *CRC Standard Mathematical Tables* contain extensive tables; the table in the CRC reference lists 600 integrals.

Example
We use the table to calculate $\displaystyle\int \frac{\sqrt{9-4x^2}}{x^2}\,dx.$

Solution
Closest to what we need is Formula 90: $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \arcsin\frac{u}{a} + C.$

We can write our integral to fit the formula by setting

$$u = 2x,\ du = 2\,dx.$$

Doing this, we have

$$\int \frac{\sqrt{9-4x^2}}{x^2}\,dx = 2\int \frac{\sqrt{9-u^2}}{u^2}\,du = 2\left[-\frac{1}{u}\sqrt{9-u^2} - \arcsin\frac{u}{3}\right] + C$$

Check this out. ↑

$$= 2\left[-\frac{1}{2x}\sqrt{9-4x^2} - \arcsin\frac{2x}{3}\right] + C.$$

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integration by Parts

(8.2.1)
$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$$

Usually we write

$$u = u(x), \qquad dv = v'(x)\,dx$$
$$du = u'(x)\,dx, \qquad v = v(x).$$

Then the formula for integration by parts reads

(8.2.2)
$$\int u\,dv = uv - \int v\,du.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integration by Parts

Example
Calculate $\int x^2 e^{-x}\,dx$

Solution
Setting $u = x^2$ and $dv = e^{-x}\,dx$, we have $du = 2x\,dx$ and $v = -e^{-x}$.
This gives

$$\int x^2 e^{-x}\,dx = \int u\,dv = uv - \int v\,du = -x^2 e^{-x} - \int -2xe^{-x}\,dx = -x^2 e^{-x} + \int 2xe^{-x}\,dx$$

We now calculate the integral on the right, again by parts. This time we set

$u = 2x$ and $dv = e^{-x}\,dx$ which gives $du = 2\,dx$ and $v = -e^{-x}$

and thus

$$\int 2xe^{-x}\,dx = \int u\,dv = uv - \int v\,du = -2xe^{-x} - \int -2e^{-x}\,dx = -2xe^{-x} + \int 2e^{-x}\,dx = -2xe^{-x} - 2e^{-x} + C$$

Combining this with our earlier calculations, we have

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C = -\left(x^2 + 2x + 2\right)e^{-x} + C$$

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integration by Parts

For definite integrals

(8.2.3)

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integration by Parts

Through integration by parts, we construct an antiderivative for the logarithm, for the arc sine, and for the arc tangent.

(8.2.4)

$$\int \ln x \, dx = x \ln x - x + C.$$

(8.2.5)

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + C.$$

(8.2.6)

$$\int \arctan x \, dx = x \arctan x - \tfrac{1}{2} \ln (1 + x^2) + C.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integration by Parts

To find the integral of the arc sine, we set

$$u = \arcsin x, \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \qquad v = x$$

This gives

$$\int \arcsin x \; dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\; dx = x \arcsin x + \sqrt{1-x^2} + C$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trigonometric Functions

Integrals of trigonometric powers and products can usually be reduced to elementary integrals by the imaginative use of the basic trigonometric identities and, here and there, some integration by parts. These are the identities that we'll rely on:

Unit circle relations

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta$$

Addition formulas

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = 1 - 2\sin^2\theta$$

Half-angle formulas

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta, \qquad \cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trigonometric Functions

Sines and Cosines

Example

Calculate $\int \sin^2 x \cos^5 x \, dx$

Solution

The relation $\cos^2 x = 1 - \sin^2 x$ enables us to express $\cos^4 x$ in terms of $\sin x$. The integrand then becomes (a polynomial in $\sin x$) $\cos x$, an expression that we can integrate by the chain rule.

$$\begin{aligned}\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \cos x \, dx \\ &= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x \, dx \\ &= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x \, dx \\ &= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C\end{aligned}$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trig Functions

Example
Calculate

$$\int \sin^2 x\, dx \quad \text{and} \quad \int \cos^2 x\, dx$$

Solution
Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

and

$$\int \cos^2 x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trigonometric Functions

Tangents and Secants

Example

Calculate

$$\int \tan^4 x \, dx$$

Solution

The relation $\tan^2 x = \sec^2 x - 1$ gives

$$\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x \sec^2 x - \sec^2 x + 1.$$

Therefore

$$\int \tan^4 x \, dx = \int \left(\tan^2 x \sec^2 x - \sec^2 x + 1 \right) dx$$

$$= \frac{1}{3} \tan^3 x - \tan x + x + C$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trig Functions

Example
Calculate $\int \sec^3 x\,dx$

Solution
The relation $\sec^2 x = \tan^2 x + 1$ gives

$$\int \sec^3 x\,dx = \int \sec x\left(\tan^2 x + 1\right)dx = \int \sec x \tan^2 x\,dx + \int \sec x\,dx$$

We know the second integral, but the first integral gives us problems. Not seeing any other way to proceed, we try integration by parts on the original integral. Setting

$$u = \sec x, \qquad dv = \sec^2 x\,dx$$
$$du = \sec x \tan x\,dx, \quad v = \tan x,$$

we have

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$\tan^2 x = \sec^2 x - 1$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Powers and Products of Trig Functions

Cotangents and Cosecants

The integrals in Examples 7–10 featured tangents and secants. In carrying out the integrations, we relied on the identity $\tan^2 x + 1 = \sec^2 x$ and in one instance resorted to integration by parts. To calculate integrals that feature cotangents and cosecants, use the identity $\cot^2 x + 1 = \csc^2 x$ and, if necessary, integration by parts.

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$

(8.4.1)

$$\text{if } F' = f, \qquad \text{then} \qquad \int f(x(u))x'(u)\,du = F(x(u)) + C$$

and

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))x'(u)\,du.$$

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Integrals that feature $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ can often be calculated by a *trigonometric substitution*. Taking $a > 0$, we proceed as follows:

for $\sqrt{a^2 - x^2}$ we set $x = a \sin u$;

for $\sqrt{a^2 + x^2}$ we set $x = a \tan u$;

for $\sqrt{x^2 - a^2}$ we set $x = a \sec u$.

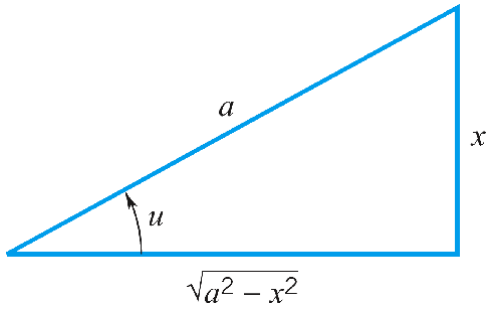
In making such substitutions, we must make clear exactly what values of u we are using.

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$

Example

To calculate $\displaystyle\int \frac{dx}{\left(a^2-x^2\right)^{3/2}}\,dx$



we note that the integral can be written

$$\int\left(\frac{1}{\sqrt{a^2-x^2}}\right)^3 dx$$

This integral features $\sqrt{a^2-x^2}$. For each x between $-a$ and a, we set

$$x = a\sin u, \quad dx = a\cos u\,du,$$

taking u between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. For such u, $\cos u > 0$ and $\sqrt{a^2-x^2} = a\cos u$

Therefore

$$\begin{aligned}\int \frac{dx}{(a^2-x^2)^{3/2}} &= \int \frac{a\cos u}{(a\cos u)^3}\,du \\ &= \frac{1}{a^2}\int \frac{1}{\cos^2 u}\,du \\ &= \frac{1}{a^2}\int \sec^2 u\,du \\ &= \frac{1}{a^2}\tan u + C = \frac{x}{a^2\sqrt{a^2-x^2}} + C.\end{aligned}$$

$\tan u = \dfrac{\sin u}{\cos u}$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$

(8.4.2)

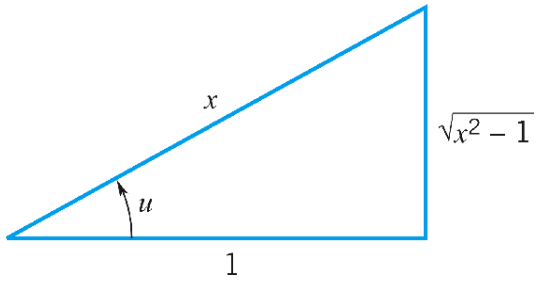
$$\int \sqrt{a^2+x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2+x^2} + \tfrac{1}{2}a^2 \ln\left(x+\sqrt{a^2+x^2}\right) + C.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$

Example
We calculate $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$



The domain of the integrand consists of two separated sets: all $x > 1$ and all $x < -1$. Both for $x > 1$ and $x < -1$, we set

$$x = \sec u, \quad dx = \sec u \tan u \, du.$$

For $x > 1$ we take u between 0 and $\frac{1}{2}\pi$; for $x < -1$ we take u between π and $\frac{3}{2}\pi$. For such u, $\tan u > 0$ and

$$\sqrt{x^2-1} = \tan u$$

Therefore,

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec u \tan u}{\tan u}\, du = \int \sec u \, du$$

$$= \ln\left|\sec u + \tan u\right| + C$$

$$= \ln\left|x + \sqrt{x^2-1}\right| + C$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Integrals Involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Trigonometric substitutions can be effective in cases where the quadratic is not under a radical sign. In particular, the reduction formula

(8.4.3)

$$\int \frac{dx}{(x^2 + a^2)^n} dx = \frac{1}{a^{2n-1}} \int \cos^{2(n-1)} u \, du$$

(a very useful little formula) can be obtained by setting $x = a \tan u$, taking u between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Rational Functions; Partial Fractions

A rational function $R(x) = P(x)/Q(x)$ is said to be *proper* if the degree of the numerator is less than the degree of the denominator. If the degree of the numerator is greater than or equal to the degree of the denominator, then the rational function is called *improper*. We will focus our attention on *proper rational functions* because any improper rational function can be written as a sum of a polynomial and a proper rational function:

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

As is shown in algebra, every proper rational function can be written as the sum of *partial fractions*, fractions of the form

(8.5.1)

$$\frac{A}{(x-\alpha)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}$$

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Rational Functions; Partial Fractions

The denominator splits into distinct linear factors.

In general, each distinct linear factor $x - \alpha$ in the denominator gives rise to a term of the form

$$\frac{A}{x - \alpha}.$$

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Rational Functions; Partial Fractions

The denominator has a repeated linear factor.

In general, each factor of the form $(x-a)^k$ in the denominator gives rise to an expression of the form

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_k}{(x-\alpha)^k}.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Rational Functions; Partial Fractions

The denominator has an irreducible quadratic factor.

In general, each irreducible quadratic factor $x^2 + \beta x + \gamma$ in the denominator gives rise to a term of the form

$$\frac{Ax + B}{x^2 + \beta x + \gamma}.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Rational Functions; Partial Fractions

The denominator has a repeated irreducible quadratic factor.

In general, each multiple irreducible quadratic factor $(x^2 + \beta x + \gamma)^k$ in the denominator gives rise to an expression of the form

$$\frac{A_1 x + B_1}{x^2 + \beta x + \gamma} + \frac{A_2 x + B_2}{(x^2 + \beta x + \gamma)^2} + \cdots + \frac{A_k x + B_k}{(x^2 + \beta x + \gamma)^k}.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Some Rationalizing Substitutions

There are integrands which are not rational functions but can be transformed into rational functions by a suitable substitution. Such substitutions are known as *rationalizing substitutions.*

Example

Find $\displaystyle\int \frac{dx}{1+\sqrt{x}}$

Solution

To rationalize the integrand, we set

$$u^2 = x, \quad 2u\,du = dx,$$

taking $u \geq 0$. Then $u = \sqrt{x}$ and

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u}{1+u}\,du \quad \xleftarrow{\text{divide}} \quad = \int \left(2 - \frac{2}{1+u}\right) du$$

$$= 2u - 2\ln(1+u) + C \quad \xleftarrow{1+u>0}$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Some Rationalizing Substitutions

Example

Find $\int \sqrt{1-e^x}\,dx$

Solution

To rationalize the integrand, we set $u = \sqrt{1-e^x}$

Then $0 \le u < 1$. To express dx in terms of u and du, we solve the equation for x:

$$u^2 = 1 - e^x, \qquad 1 - u^2 = e^x, \qquad \ln(1-u^2) = x, \qquad -\frac{2u}{1-u^2}\,du = dx.$$

$$\int \sqrt{1-e^x}\,dx = \int u\left(-\frac{2u}{1-u^2}\right)du$$

$$= \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$

divide; then use partial fractions

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C$$

$$= 2\sqrt{1-e^x} + \ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right| + C. \quad \square$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

To evaluate a definite integral of a continuous function by the formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

we must be able to find an antiderivative F and we must to able to evaluate this antiderivative both at a and at b. When this is not feasible, the method fails.

The method fails even for such simple-looking integrals as

$$\int_0^1 \sqrt{x}\sin x\,dx \quad \text{or} \quad \int_0^1 e^{-x^2}\,dx$$

In the following slides we will illustrate some simple numerical methods for estimating definite integrals—methods that you can use whether or not you can find an antiderivative. All the methods described involve only simple arithmetic and are ideally suited to the computer.

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

The region Ω_1 pictured in Figure 8.7.1, can be approximated in many ways:

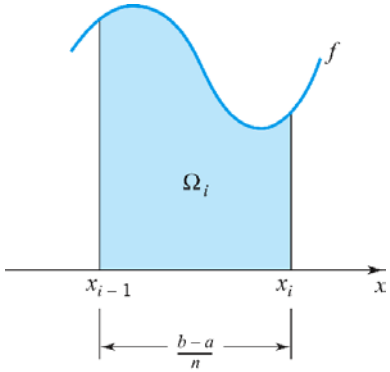


Figure 8.7.1

The left-endpoint rectangle:

$$\text{area} \cong f(x_{i-1})\Delta x_i = f(x_{i-1})\left(\frac{b-a}{n}\right)$$

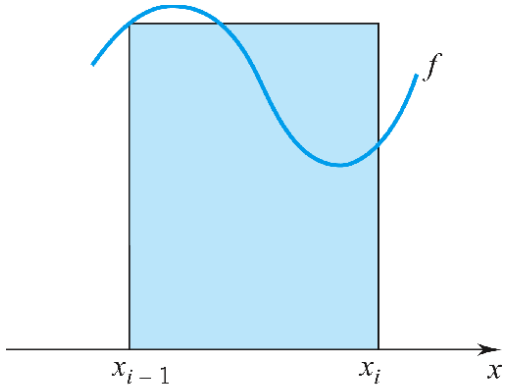


Figure 8.7.2

The approximation to Ω_1 just considered yields the following estimate for

$$\int_a^b f(x)\,dx:$$

The left-endpoint estimate:

$$L_n = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots + f(x_{n-1})].$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

The region Ω_1 pictured in Figure 8.7.1, can be approximated in many ways:

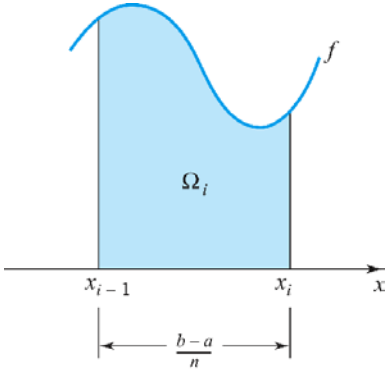


Figure 8.7.1

The right-endpoint rectangle:

$$\text{area} \cong f(x_i)\Delta x_i = f(x_i)\left(\frac{b-a}{n}\right).$$

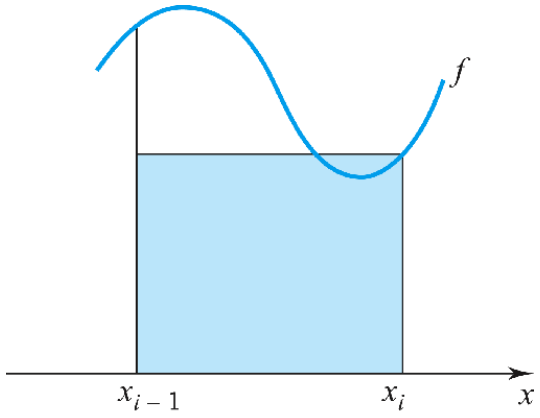


Figure 8.7.3

The approximation to Ω_1 just considered yields the following estimate for

$$\int_a^b f(x)\,dx:$$

The right-endpoint estimate:

$$R_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)].$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

The region Ω_1 pictured in Figure 8.7.1, can be approximated in many ways.

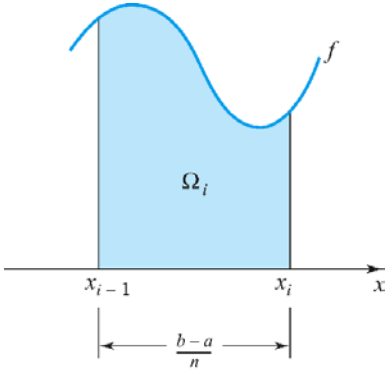


Figure 8.7.1

The midpoint rectangle:

$$\text{area} \cong f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i = f\left(\frac{x_{i-1}+x_i}{2}\right)\left(\frac{b-a}{n}\right).$$

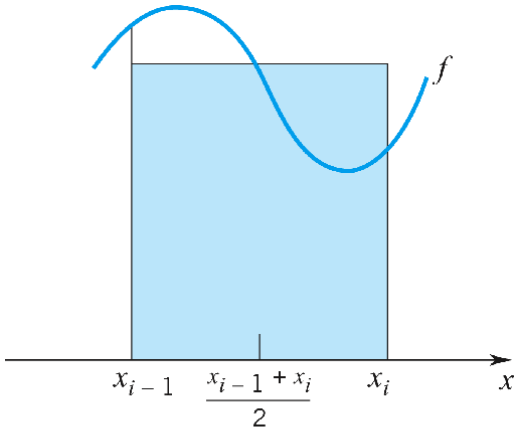


Figure 8.7.4

The approximation to Ω_1 just considered yields the following estimate for

$$\int_a^b f(x)\,dx:$$

The midpoint estimate:

$$M_n = \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right].$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

By a trapezoid:

$$\text{area} \cong \frac{1}{2}[f(x_{i-1}) + f(x_i)]\Delta x_i = \frac{1}{2}[f(x_{i-1}) + f(x_i)]\left(\frac{b-a}{n}\right).$$

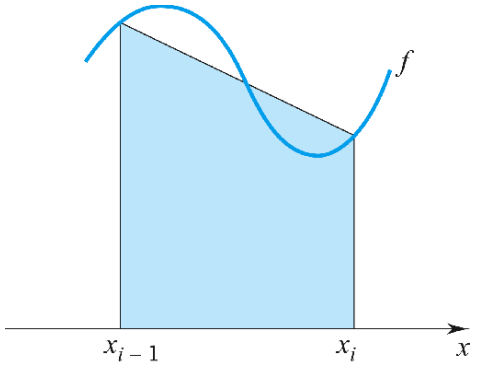


Figure 8.7.5

The approximation to Ω_1 just considered yields the following estimate for

$$\int_a^b f(x)\,dx:$$

The trapezoidal estimate (*trapezoidal rule*):

$$T_n = \frac{b-a}{n}\left[\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right]$$

$$= \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

By a parabolic region (Figure 8.7.6): take the parabola $y = Ax2 + Bx + C$ that passes through the three points indicated.

$$\text{area} \cong \frac{1}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right]\Delta x_i$$

$$= \left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right]\left(\frac{b-a}{6n}\right).$$

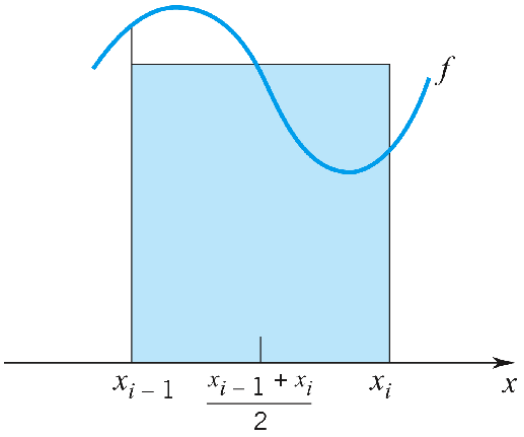


Figure 8.7.4

The approximation to Ω_1 just considered yields the following estimate for

$$\int_a^b f(x)\,dx:$$

The parabolic estimate (*Simpson's rule*):

$$S_n = \frac{b-a}{6n}\left\{f(x_0) + f(x_n) + 2[f(x_1) + \cdots + f(x_{n-1})] + 4\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\right\}.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

Example

Find the approximate value of $\int_0^3 \sqrt{4+x^3}\,dx$ by the trapezoidal rule. Take $n = 6$.

Solution

Each subinterval has length $\dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$

The partition points are

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2, \quad x_5 = \frac{5}{2}, \quad x_6 = 3$$

Then $T_6 \cong \frac{1}{4}[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)],$

with $f(x) = \sqrt{4+x^3}$. Using a calculator and rounding off to three decimal places, we have

$$f(0) = 2.000, \quad f\left(\tfrac{1}{2}\right) \cong 2.031, \quad f(1) \cong 2.236, \quad f\left(\tfrac{3}{2}\right) \cong 2.716,$$
$$f(2) \cong 3.464, \quad f\left(\tfrac{5}{2}\right) \cong 4.430, \quad f(3) \cong 5.568.$$

Thus

$$T_6 \cong \tfrac{1}{4}(2.000 + 4.062 + 4.472 + 5.432 + 6.928 + 8.860 + 5.568) \cong 9.331.$$

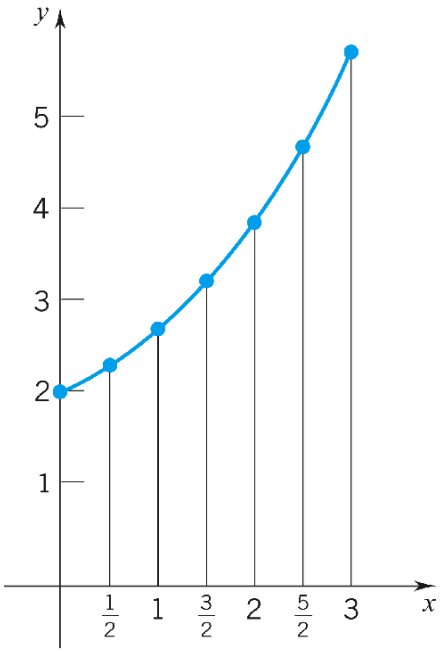


Figure 8.7.8

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

Error Estimates
A numerical estimate is useful only to the extent that we can gauge its accuracy.

Theoretical error: Error inherent in the method we use

Round-off Error: Error that accumulates from rounding off the decimals that arise during the course of computation

Consider a function f continuous and increasing on $[a, b]$.
Subdivide $[a, b]$ into n nonoverlapping intervals, each of length $(b - a)/n$.
Estimate

$$\int_a^b f(x)\,dx$$

by the left-endpoint method.

What is the theoretical error?

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

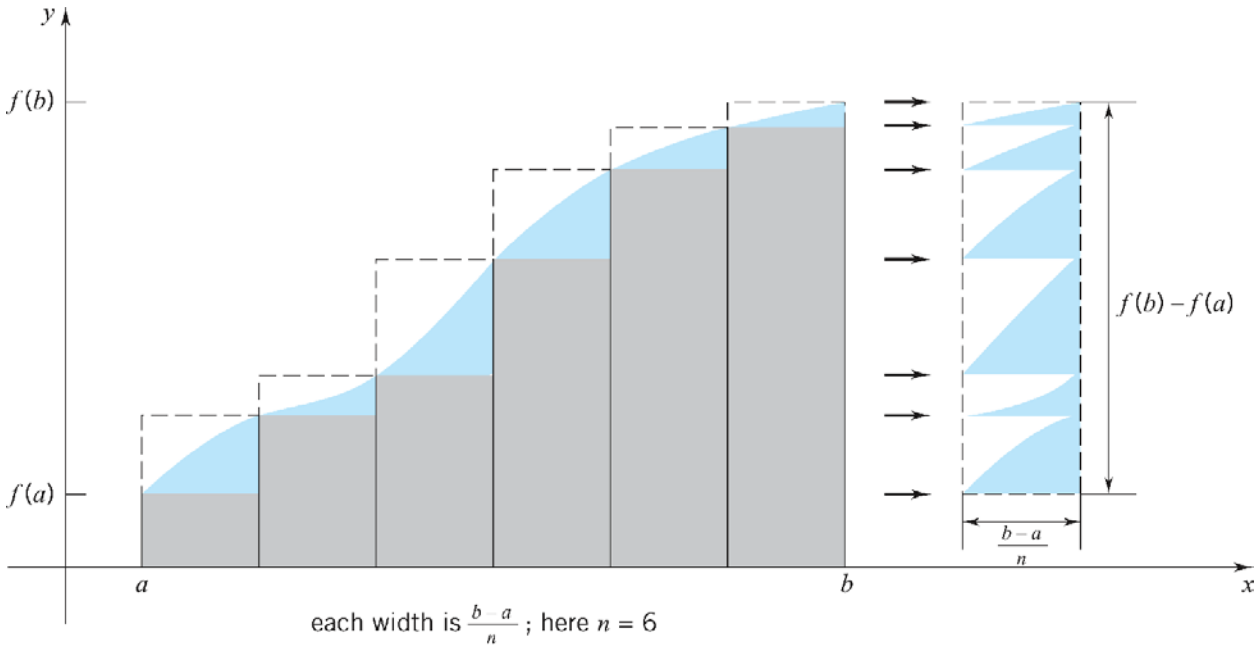


Figure 8.7.9

The theoretical error does not exceed

$$\left[f(b) - f(a) \right] \left(\frac{b-a}{n} \right)$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

$$E_n^T = \int_a^b f(x)\,dx - T_n$$

can be written

(8.7.1)

$$E_n^T = -\frac{(b-a)^3}{12\,n^2} f''(c),$$

where c is some number between a and b. Usually we cannot pinpoint c any further. However, if f'' is bounded on $[a, b]$, say $|f''(x)| \le M$ for $a \le x \le b$, then

(8.7.2)

$$|E_n^T| \le \frac{(b-a)^3}{12\,n^2} M.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

$$E_n^S = \int_a^b f(x)\,dx - S_n$$

can be written

(8.7.3)

$$E_n^S = -\frac{(b-a)^5}{2880n^4} f^{(4)}(c),$$

where, as before, c is some number between a and b. Whereas (8.7.1) varies as $1/n^2$, this quantity varies as $1/n^4$. Thus, for comparable n, we can expect greater accuracy from Simpson's rule. In addition, if we assume that $f^{(4)}(x)$ is bounded on $[a, b]$, say $|f^{(4)}(x)| \le M$ for $a \le x \le b$, then

(8.7.4)

$$|E_n^S| \le \frac{(b-a)^5}{2880n^4} M.$$

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.

Numerical Integration

To estimate

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

by the trapezoidal rule with theoretical error less than 0.00005, we needed to subdivide the interval [1, 2] into at least fifty-eight subintervals of equal length. To achieve the same degree of accuracy with Simpson's rule, we need only four subintervals:

$$\text{for } f(x) = 1/x, \quad f^{(4)}(x) = 24/x^4.$$

Therefore $|f^{(4)}(x)| \le 24$ for all $x \in [1, 2]$ and

$$\left|E_n^S\right| \le \frac{(2-1)^5}{2880n^4}\,24 = \frac{1}{120n^4}$$

This quantity is less than 0.00005 provided only that $n^4 > 167$. This condition is met by $n = 4$.

Main Menu

Salas, Hille, Etgen Calculus: One and Several Variables
Copyright 2007 © John Wiley & Sons, Inc. All rights reserved.