The Existence of Hamiltonian Stationary Lagrangians

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Riemannian structure | Symplectic structure

g | (N^{2n}, ω)

minimal submfd | Lag submfd

$H=0$ | L^n. $\omega|_{L^n}=0$

ex: Kähler mfd

- Lag & minimal $\leftrightarrow$ sLag in Calabi-Yau
- On Lag., $d\alpha_H = \mathrm{Ric}|_L$ $\quad \alpha_H = \omega(H,\cdot)$

$\hookrightarrow$ Ricci form

- Existence is a big problem
- minimize area among Lags

  to have better E-L eqs

  further restrict to Hamiltonian variation
- Hamiltonian stationary Lag (HSL) (OH)

Critical pts of area among Hamiltonian variations

$$\Leftrightarrow \quad 0 = \left.\frac{dA_t}{dt}\right|_{t=0} = -\int_L \langle H, J\nabla f\rangle \, dVol_L \quad \forall \text{ fun } f \text{ on } L$$

- models for incompressible elastic theories
- Important in Schoen & Wolfson's approach :
- E-L eq $\boxed{d^* \alpha_H = 0}$
- can be defined in a symplectic mfd with compatible metrics

$\omega \to J \qquad \to g(u,v) = \omega(u, Jv)$

$J^2 = -id$

$\omega(Ju, Jv) = \omega(u, v)$

- Existence of HSL

① in $\mathbb{C}^n$ ($\exists$ cpt examples)

② Helein & Romon found Weierstass-type representation for HSLT in $\mathbb{C}^2$ & $\mathbb{CP}^2$

③ some examplee in special homogenoue spacee

④ existence in general Kahler mfd unkown

Thm 1 (Joyce, L– & Schoen, to appear Amer. J. of Math.

(M, ω, g) a $\boxed{\text{cpt}}$. symplectic mfd. g a compatible metric

$L^n \subset \mathbb{C}^n$. a rigid cpt. HSL.

$\Rightarrow$ ① $\forall$ small $t > 0$. $\exists$ HSL L' contained in a ball

of radius t at some pt $p \in (M, \omega, g)$

$L' \overset{\text{diff}}{\approx} L$, a perturbation of $t L$

② If L is H-stable. can take L' H-stable

Remark

- a HSL is called H-stable if $\frac{d^2A}{dt^2}\Big|_{t=0} \geq 0$
  w.r.t H-deformation

- The linearlization of the E-L op

$$\mathcal{L}f = -\frac{d}{ds}(d^*\alpha_{H_s})\Big|_{s=0}$$

$f \longmapsto F \longrightarrow J\nabla F \longrightarrow \varphi_s$ symplectic $\longrightarrow L_s = \varphi_s(L)$

- In Kähler,

$$\mathcal{L}f = \Delta^2 f + d^*\alpha_{\mathrm{Ric}^\perp(J\nabla f)} - 2d^*\alpha_{B(JH,\nabla f)} - JH(JH(f)).$$

$$\alpha_v(\cdot) = \omega(v, \cdot)$$

- On a HS in Kähler.

$$\left.\frac{d^2 A(t)}{dt^2}\right|_{t=0} = \int_L \langle \mathcal{L}f, f \rangle$$

- Now suppose L HS in $\mathbb{C}^n$

$U(n) \curvearrowright \mathbb{C}^n$ acts on $\mathbb{C}^n$ preserving g_0, ω_0, J_0

$\Rightarrow$ {The corresponding H-func of $U(n) \curvearrowright \mathbb{C}^n$}

Moment map

$\subset \operatorname{Ker} \mathcal{L}$ $(*)$

L is called H-rigid if "$=$" in $*$ $\Rightarrow L$ conn

Examples for H-stable & H-rigid

ⓐ $T^n_{a_1,\ldots,a_n} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : |z_j| = a_j,\ j = 1,\ldots,n\}.$

H-rigid & H-stable HSL (OH. 1993. Math. Z.)

ⓑ other examples. (Armazaya & Ohnita)

For Thm 1

Advantage

disadvantage

Thm 2 (L–, 2010)

(M^n, ω, g), kähler, . $\mathcal{U}$: $U(n)$-frame bundle

$$F_{a_1 \cdots a_n}(P, [v]) = \sum_{i=1}^{n} a_i^2 R_{i\bar{i}i\bar{i}}(P) : \mathcal{U}/T^n \longmapsto \mathbb{R} \qquad a_i > 0$$

Suppose $(P_0, [v_0])$ is a non-deg critical of $F_{a_1 \cdots a_n}$.

Then for t small, $\exists$ smooth $(P(t), [v(t)]) \in \mathcal{U}/T^n$

and embedded HSLT $\simeq T^n_{(ta_1, \cdots ta_n)}$ center at $p(t)$

& $d\big((p(t), [v(t)]), (P_0, [v_0])\big) \leq ct^2$ ($\Rightarrow$ no intersection)

<u>RmK</u> Butscher & Corvino, different approach, & a different condition for $n=2$.

- An analogue to CMC hypersurfaces in Riem by Ye.

Ideas of our proof

Step 1: Construct approximate sols Darboux coord.

Step 2: Perturb to real solutions

Singular perturbation

main difficulties in step 2

Linearlized op. has approximate kernels

need 2 substeps to complete

① solve the problem ⊥ approximate kernels
to get better approximate sols
(perturb to a better approximation)

② extra condition to obtain exact sols

Step 1 : Rederive Darboux coordinates

Prop: (M, ω, g) Kähler $\Rightarrow \exists\ \Upsilon_{p,v} : B_\varepsilon(0) \subset \mathbb{C}^n \longmapsto M$

embedding & smooth on $(p, v) \in U$. $\ni$

(i) $\Upsilon_{p,v}(0) = p$ *and* $\mathrm{d}\Upsilon_{p,v}|_0 = v : \mathbb{C}^n \to T_pM$;

(ii) $\Upsilon_{p,v\circ\gamma} \equiv \Upsilon_{p,v} \circ \gamma$ *for all* $\gamma \in \mathrm{U}(n)$;

(iii) $\Upsilon^*_{p,v}(\omega) = \omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$; *and*

$g_0 = \sum |dz_j|^2$

(iv) $\Upsilon^*_{p,v}(g) = g_0 + \frac{1}{2}\sum \mathrm{Re}\big(R_{i\bar{j}k\bar{l}}(p) z^i z^k \mathrm{d}\bar{z}^j \mathrm{d}\bar{z}^l\big) + \frac{1}{5}\sum \mathrm{Re}\big(R_{i\bar{j}k\bar{l},m}(p) z^i z^k z^m \mathrm{d}\bar{z}^j \mathrm{d}\bar{z}^l\big)$

$+ \ \frac{2}{5}\sum \mathrm{Re}\big(R_{i\bar{j}k\bar{l},\bar{m}}(p) z^i z^k \bar{z}^m \mathrm{d}\bar{z}^j \mathrm{d}\bar{z}^l\big) + O\big(|z|^4\big)$

When (M, ω, g) symplectic, $\Upsilon^*_{p,v}(g) = g_0 + O(|z|)$

Posit HSL $tL \subseteq (\mathbb{C}^n, \omega_0)$ in such coordinates

Image Lag, but not HS in (M, ω, g)

Rmk : B.C. : hol coordinates & potential of ω

- Use Lag Neighborhood Thm to deform the Lag

$$\Gamma_{df} \subset (T^*L, \hat{\omega}) \xrightarrow{\Phi} L \subset (\mathbb{C}^n, \omega_0) \xrightarrow[t]{\text{scaling}} (\mathbb{C}^n, \omega_0) \xrightarrow{\gamma_{p,v}} (M, \omega)$$

$\Phi(\Gamma_{df}) \subset B_R$ $\qquad t\,\Phi(\Gamma_{df}) \subset B_{tR}$

$dp^i \wedge dq^i = d(p^i dq^i)$

assume $\int f \, dV = 0$

- $\gamma_{p,v} \circ t \circ \Phi(\Gamma_{df}) = L^{t,f}_{p,v}$ Lag in (M, ω)

  given Hamiltonian deformation of $\gamma_{p,v} \circ t(L)$

Denote $g^t_{p,v} = t^{-2}(\gamma_{p,v} \circ t)^* g = \begin{cases} g_0 + O(t^2|z|^2) & \text{Kahler} \\ g_0 + O(t|z|) & \text{symplectic} \end{cases}$

$\Rightarrow t^{-2}(\gamma_{p,v} \circ t)^*(\omega) = \omega_0$

$\| g^t_{p,v} - g_0 \|_{C^0} \leq C_0 t^2$, (Ct_0) $\quad \| \partial^k g^t_{p,v} \|_{C^0} \leq C_k t^{k+1}$ $(C_k t^k)$

work on $(B_R, g^t_{p,v}, \omega_0)$ instead

fixed object $\Phi(\Gamma_{df}) \subset B_R \subset \mathbb{C}^n$.

w.r.t different metrics $g^t_{p,v}$

Step 2

Consider

$$F^t_{p,v}(f) = \mathrm{Vol}_{g^t_{p,v}}(\Phi(\Gamma_{df})) = t^{-n}\mathrm{Vol}_g(L^{t,f}_{p,v})$$

$$= \int_{\Phi(\Gamma_{df})} dV_{g^t_{p,v}|_{\Phi(\Gamma_{df})}} = \int_L (\Phi_f)^*(dV_{g^t_{p,v}|_{\Phi(\Gamma_{df})}})$$

$$= \int_L G^t_{p,v}(q, df|_q, \nabla df|_q)\, dV_{g_0|_L}.$$

& $F_0(f)$. w.r.t g_0

E-L ops $P^t_{p.v}(f)$ & $P_0(f)$, goal: find. zeros of $P^t_{p.v}$

Linealization $\mathcal{L}^t_{p,v}(f)$, & $\mathcal{L}(f)$ w.r.t dV_0

Prop: $k \geq 0$ integer, $\gamma \in (0,1)$, $\forall\ \zeta > 0$ $\exists\ t_0$, $\ni$ $\forall\ t < t_0$

$$\|P^t_{p,v}(f) - P_0(f)\|_{C^{k,\gamma}} \leqslant \zeta \quad \textit{and} \quad \|\mathcal{L}^t_{p,v}(f) - \mathcal{L}(f)\|_{C^{k,\gamma}} \leqslant \zeta \|f\|_{C^{k+4,\gamma}},$$

① **Lemma** $\exists\, f^t_{p,v} \in C^\infty(L) \cap (\operatorname{Ker} \mathcal{L})^\perp$ with $P^t_{p,v}(f^t_{p,v}) \in \operatorname{Ker} \mathcal{L}$

$f^t_{p,v}$ is unique for $\| f^t_{p,v} \|_{C^{4,\gamma}}$ small

and depends smoothly on $(p,v) \in U$

ⓐ work on space $\perp \operatorname{Ker} \mathcal{L}$

ⓑ By IFT $\qquad \| f^t_{p,v} \| \leq C t^2$ for Kähler

ⓒ smoothness & smooth dependent

Note : the same space not depending on (t, p, v)

different metrics & ops

Denote $L^t_{p,v} = L^{t, f^t_{p,v}}_{p,v}$,

WOLG, We can assume the invariant group G for L is in $U(n)$

$\Rightarrow L^t_{p,v}$ is G-inv.

(2) Define $H^t : (p,v) \in U \longmapsto P^t_{p,v}(f^t_{p,v}) \in \mathrm{Ker}\, \mathcal{L}$

reduce ∞ dim to finite dim

$K^t : (p,v) \in U \longmapsto t^{-n} \mathrm{Vol}_g(L^t_{p,v}) \in \mathbb{R}$

Lemma: With suitable identification

$H^t = dK^t$ (need L rigid)

Pf: (i) $dK^t \in T^*_{(p,v)} U \cong (\mathfrak{u}(n) \oplus \mathbb{C}^n)^*$

$\because L^t_{p,v} = L^t_{p,v\circ\gamma}$ for $\gamma \in G$ (Lie alg $\mathfrak{g}$)

$\Rightarrow dK^t|_{(p,v)}$ lies in the annihilator $\mathfrak{g}^\circ$ of $\mathfrak{g}$

$\dim \mathfrak{g}^\circ = n^2 + 2n - \dim G$

(ii) $H^t : (p,v) \in U \longmapsto P^t_{p,v}(L^t_{p,v}) \in \operatorname{Ker} \mathcal{L}$

rigid $\Rightarrow \dim \operatorname{Ker} \mathcal{L} = n^2 + 2n + 1 - \dim G$

$P^t_{p,v}(f^t_{p,v}) = d^*\eta$ for some 1-form $\Rightarrow \int P^t_{p,v}(f^t_{p,v}) = 0$

$\therefore$ H^t maps U to the subspace $\{f \in \operatorname{Ker} \mathcal{L},\ \int_L f\, dV = 0\}$

$\dim \{f \in \operatorname{Ker} \mathcal{L},\ \int_L f\, dV = 0\} = \dim \mathfrak{g}^0$

- When M cpt sym. $\Rightarrow U$ cpt

K^t is a smooth fun on $U \Rightarrow K^t$ has critical pts

$dK^t|_{(p,v)} = 0 \Leftrightarrow 0 = H^t(p,v) = P^t_{p,v}(f^t_{p,v})$

If L is H-stable, choose (p,v) local min of K^t, then $L^t_{p,v}$ is H-stable in (M,ω,g)

- Do not know where the critical pts are and may not smoothly depend on K^t

need some kind of non-deg critical pts

🚩 For Thm 2 need detailed estimates.

Remember: reduced to find critical pts of K^t

When $L = T^n_{a_1 \cdots a_n} = \{(a_1 e^{i\theta_1}, \cdots a_n e^{i\theta_n})\}$,

Two proofs: ①

②

Claim: $$K^t(p,v) = \left(1 - \frac{1}{4}t^2 \sum_{i=1}^{n} a_i^2 R_{i\bar{i}i\bar{i}}(p)\right) \mathrm{Vol}_{g_0}(T^n_{a_1,\dots,a_n}) + O(t^4).$$

K^t & $\sum_{i=1}^{n} a_i^2 R_{i\bar{i}i\bar{i}}(p)$ inv under T^n inv. group

$\Rightarrow$ a map from U/T^n to $\mathbb{R}$

$\because$ $(p_0, [v_0])$ a non-deg critical of $F_{a_1,\dots,a_n}$

Implicit fun Thm (For dK at $t=0$)

$\Rightarrow \exists$ a smooth family $(p(t), [v(t)]) \in U/T^n$

$\ni$ $(p(t), [v(t)])$ a critical pt of K^t

$(P(0), [v(0)]) = (P_0, v_0)$ & $d\left[(p(t)[v(t)]), (P_0, [v_0])\right] \leq ct^2$

$\because$ the HSLT for t has radi $ta_1, \cdots ta_n$

$\therefore$ The family does not intersect.

Proof of the claim

① $K^t(p,v) = F^t_{p,v}(0) + O(t^4).$ for $\forall$ HS

② $F^t_{p,v}(0) = \left(1 - \frac{1}{4} t^2 \sum a_i^2 R_{i\bar{i}i\bar{i}}(P)\right) F(0) + O(t^4)$

for T^n

① do the expansion of vol elts in t

If $h = h_0+t^2h_2+t^3h_3+O(t^4)$, then

$$\sqrt{\det(h)} = \sqrt{\det(h_0)}\left(1+\frac{1}{2}t^2\,\mathrm{Tr}(h_0^{-1}h_2)+\frac{1}{2}t^3\,\mathrm{Tr}(h_0^{-1}h_3)+O(t^4)\right),$$

$$\|f^t_{p,v}\|_{k,r} \le c\,t^2$$

On induced metric h

$\Rightarrow$ the contribution of $f^t_{p,v}$ & $g^t_{p,v}$ separate for order $< t^4$

$$\begin{aligned} g^t_{p,v} =&g_0+\frac{t^2}{2}\sum_{i,j,k,l}\mathrm{Re}(R_{i\bar{j}k\bar{l}}(p)z^iz^k\mathrm{d}\bar{z}^j\mathrm{d}\bar{z}^l)\\ &+\frac{t^3}{5}\sum_{i,j,k,l,m}\mathrm{Re}(R_{i\bar{j}k\bar{l},m}(p)z^iz^kz^m\mathrm{d}\bar{z}^j\mathrm{d}\bar{z}^l)\\ &+\frac{2t^3}{5}\sum_{i,j,k,l,m}\mathrm{Re}(R_{i\bar{j}k\bar{l},\bar{m}}(p)z^iz^k\bar{z}^m\mathrm{d}\bar{z}^j\mathrm{d}\bar{z}^l)+O(t^4|z|^4). \end{aligned}$$

$$
\begin{aligned}
K^t(p,v) = F^t_{p,v}(f^t_{p,v}) &= \mathrm{Vol}_{g^t_{p,v}}\,\Phi(\Gamma_{\mathrm{d}f^t_{p,v}}) \\
&= \int_{T^n_{a_1,\dots,a_n}} \left(1 + \frac{1}{2}t^2\,\mathrm{Tr}(h_0^{-1}h_2) + \frac{1}{2}t^3\,\mathrm{Tr}(h_0^{-1}h_3) + O(t^4)\right)\mathrm{d}V_0 \\
&= F_0(f^t_{p,v}) + F^t_{p,v}(0) - F_0(0) + O(t^4).
\end{aligned}
$$

$$
F_0(f^t_{p,v}) = F_0(0) + \frac{\mathrm{d}}{\mathrm{d}s}F_0(sf^t_{p,v})\Big|_{s=0} + O(t^4) \overset{\text{(i) HS}}{=} F_0(0) + O(t^4).
$$

$$
\Rightarrow \quad K^t(p,v) = \underline{F^t_{p,v}}(0) + O(t^4)
$$

② In polar coordinates

$$
T^n_{a_1,\dots,a_n} = \{(a_1 e^{\sqrt{-1}\theta_1}, \dots, a_n e^{\sqrt{-1}\theta_n}) \in \mathbb{C}^n : \theta_i \in [0, 2\pi), i = 1, \dots, n\},
$$

$$
h^t_{p,v} = \sum a_i^2 \mathrm{d}\theta_i^2 - \sum a_i a_j (t^2\,\mathrm{Re}\,A_{ij} + t^3\,\mathrm{Re}\,C_{ij})\mathrm{d}\theta_i \mathrm{d}\theta_j + O(t^4).
$$

$$F^t_{p,v}(0) = F_0(0) - \frac{1}{2}\int_{T^n_{a_1,\dots,a_n}} \sum_{i=1}^{n} (t^2 \operatorname{Re} A_{ii} + t^3 \operatorname{Re} C_{ii})\, dV_0 + O(t^4),$$

$$A_{ij} = A_{ji} = \frac{1}{2}\sum_{p,q} R_{p\bar{i}q\bar{j}}(p) r_p r_q e^{\sqrt{-1}(\theta_p+\theta_q-\theta_i-\theta_j)},$$

$$C_{ij} = C_{ji} = \frac{1}{5}\sum_{p,q,m} R_{p\bar{i}q\bar{j},m}(p) r_p r_q r_m e^{\sqrt{-1}(\theta_p+\theta_q-\theta_i-\theta_j+\theta_m)}$$

$$+ \frac{2}{5}\sum_{p,q,m} R_{p\bar{i}q\bar{j},\bar{m}}(p) r_p r_q r_m e^{\sqrt{-1}(\theta_p+\theta_q-\theta_i-\theta_j-\theta_m)}.$$

free variables are θ's

the integration of sin or cos in $T^n = 0$

$$\Rightarrow F^t_{p,v}(0) = \left(1 - \frac{1}{4}t^2 \sum a_i^2 R_{i\bar{i}i\bar{i}}(p)\right) F(0) + O(t^4)$$

① + ② $\Rightarrow$ claim.

~ The End ~

Thank you !!