# The mean curvature flow of compact submanifolds in higher codimension

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# References


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- M.-P. Tsui; M.-T. Wang, *Mean curvature flows and isotopy of maps between spheres.* Comm. Pure Appl. Math. **57** (2004), no. 8, 1110–1126.

# Outline

1. Preliminaries.
   - Mean curvature flow.
   - Main results.
   - Parallel $n$-form.
2. Proof of theorem.
   - Long time existence.
   - Convergence.
3. Applications.

# Mean curvature flow (MCF) in Riemannian manifolds.

- $(N_1^n, g)$ and $(N_2^m, h)$: compact Riemannian manifolds.
- $f : N_1 \to N_2$ a smooth map. Denote $\Sigma = (x, f(x))$: the graph of $f$.
- $\Sigma$: embedded submfd in $M = N_1 \times N_2$ with $F = \text{id.} \times f : N_1 \to M$.
- MCF of $\Sigma$ is a smooth family $F_t : N_1 \to M$ satisfying

$$\begin{cases} \left(\dfrac{\partial F_t(x)}{\partial t}\right)^{\perp} = H(x,t) \\ F_0(N_1) = \Sigma \end{cases}$$

  $H$: mean curvature vector of $F_t(N_1) = \Sigma_t$.

  $(\cdot)^{\perp}$: projection onto the normal bundle $N\Sigma_t$ of $\Sigma_t$.
- By standard PDE theory, the flow has short time existence.

## Results

**Theorem** (K.-W. Lee, Y.-I. Lee)

*Let* $f : (N_1, g, K_{N_1} \geq k_1) \to (N_2, h, K_{N_2} \leq k_2)$.

*Suppose either* $k_1 \geq 0, k_2 \leq 0$*, or* $k_1 \geq k_2 > 0$.

*If* $\frac{\det((g+f^*h)_{ij})}{\det(g_{ij})} < 4$*, or* $f$*: area decreasing map, then*

(i) *The graph of* $f$ *is preserved along MCF; long time existence.*

(ii) *If* $k_1 > 0$ *(locally symmetric), then* $f$ *converges to a constant map.*

**Remark**

In Tsui and Wang's paper, constant curvature, $k_1 \geq |k_2|$, and $\det < 2$.

# Parallel $n$-form $\Omega$ on $M$ ($\nabla^M\Omega = 0$); evolution equation

- Choose o.n. frames $\{e_i\}_{i=1}^n$ for $T\Sigma_t$ and $\{e_\alpha\}_{\alpha=n+1}^{n+m}$ on $N\Sigma_t$.
- $\Omega_{1\cdots n} = \Omega(e_1, \ldots, e_n)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\Omega_{1\cdots n} =& \Delta\Omega_{1\cdots n} + \Omega_{1\cdots n}\left(\sum_{\alpha,i,k}(h^\alpha_{ik})^2\right) \\ & - 2\sum_{\alpha<\beta,k}\left(\Omega_{\alpha\beta 3\cdots n}h^\alpha_{1k}h^\beta_{2k} + \cdots + \Omega_{1\cdots(n-2)\alpha\beta}h^\alpha_{(n-1)k}h^\beta_{nk}\right) \\ & - \sum_{\alpha,k}\left(\Omega_{\alpha 2\cdots n}R_{\alpha kk1} + \cdots + \Omega_{1\cdots(n-1)\alpha}R_{\alpha kkn}\right)\end{aligned}$$

$\Delta$: time-dependent Laplacian on $\Sigma_t$. $h^\alpha_{ij} = \langle \nabla^M_{e_i} e_j, e_\alpha\rangle$.
$R$: the curvature tensor of $M = N_1 \times N_2$ with $g + h$.

# A special parallel $n$-form

- Since $M = N_1 \times N_2$, the volume form $\Omega_1$ of $N_1$ can be extended as a parallel $n$-form on $M$.
- At $p$ on $\Sigma_t$, we have $*\Omega = \Omega_1(e_1, \ldots, e_n) = \Omega_1(\pi_1(e_1), \ldots, \pi_1(e_n))$. Jacobian of the projection from $T_p\Sigma_t$ to $T_{\pi_1(p)}N_1$.
- By the implicit function theorem, we know $*\Omega > 0$ near $p \Longleftrightarrow \Sigma_t$ is locally a graph over $N_1$ near $p$.

# Singular value decomposition theorem

## Theorem

$$[A]_{m\times n} = [U]_{m\times m}[\Lambda]_{m\times n}\left[V^T\right]_{n\times n}.$$

- $U, V$: *orthogonal.*
- $\Lambda$: *diagonal, and*
  - $\Lambda_{ii} = \lambda_i$, $\lambda_1 \geq \cdots \geq \lambda_r > 0$, $r =$ *rank* $A$.
  - $\Lambda_{ii} = 0 \quad \forall\, i = r+1, \ldots, \min\{m, n\}$.

## Remark

$$A = U\Lambda V^T \Leftrightarrow AV = U\Lambda \Leftrightarrow Av_i = \lambda_i u_i.$$

# Singular value decomposition theorem (continued)

- Apply SVD to $\mathrm{d}f_t : T_{\pi_1(p)}N_1 \to T_{\pi_2(p)}N_2$, $\exists$ o.n. basis $\{a_i\}_{i=1}^n$ for $T_{\pi_1(p)}N_1$ and $\{a_\alpha\}_{\alpha=n+1}^{n+m}$ for $T_{\pi_2(p)}N_2$ such that

$$\mathrm{d}f_t(a_i) = \lambda_i a_{n+i} \text{ for } 1 \le i \le r, \text{ and } \mathrm{d}f_t(a_i) = 0 \text{ for } r \le i \le n.$$

- Get special o.n. bases $\{E_i\}_{i=1}^n$ on $T_p\Sigma_t$ and $\{E_\alpha\}_{\alpha=n+1}^{n+m}$ on $N_p\Sigma_t$:

$$E_i = \begin{cases} \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i}) & \text{if } 1 \le i \le r \\ a_i & \text{if } r+1 \le i \le n, \end{cases}$$

$$E_{n+q} = \begin{cases} \frac{1}{\sqrt{1+\lambda_q^2}}(a_{n+q} - \lambda_q a_q) & \text{if } 1 \le q \le r \\ a_{n+q} & \text{if } r+1 \le q \le m, \end{cases}$$

- Thus,

$$*\Omega = \Omega_1(\pi_1(E_1), \ldots, \pi_1(E_n)) = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}.$$

## Lemma (Evolution Equation for $*\Omega$, M.-T. Wang)

*If the MCF of $\Sigma$ is a graph over $N_1$, then $*\Omega$ satisfies:*

$$\begin{aligned}\frac{\partial}{\partial t} *\Omega =& \Delta *\Omega + *\Omega|A|^2 + *\Omega\left\{2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i} - 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+i}h_{jk}^{n+j}\right\}\\ &+ *\Omega\sum_{i,k}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\langle R_1(a_k,a_i)a_k,a_i\rangle\right.\\ &\left.\qquad - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\langle R_2(a_{n+k},a_{n+i})a_{n+k},a_{n+i}\rangle\right)\end{aligned}$$

*where $|A|^2$: norm square of the second fundamental form, and*
*$R_1, R_2$: curvature tensors on $(N_1, g), (N_2, h)$, respectively.*

Lemma (Evolution Eqn. for $\ln *\Omega$, M.-P. Tsui; M.-T. Wang.)

*The evolution equation can be rewritten as the form:*

$$\begin{aligned}\frac{\partial}{\partial t}\ln *\Omega =&\Delta \ln *\Omega + |A|^2 + \sum_{i,k}\lambda_i^2\left(h_{ik}^{n+i}\right)^2 + 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i}\\ &+\sum_{i,k}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\langle R_1(a_k,a_i)a_k,a_i\rangle\right.\\ &\left.\qquad -\frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\langle R_2(a_{n+k},a_{n+i})a_{n+k},a_{n+i}\rangle\right)\end{aligned}$$

# Results

## Theorem (K.-W. Lee, Y.-I. Lee)

*Let* $f : (N_1, g, K_{N_1} \geq k_1) \to (N_2, h, K_{N_2} \leq k_2)$.

*Suppose either* $k_1 \geq 0, k_2 \leq 0$, *or* $k_1 \geq k_2 > 0$.

*If* $\frac{\det((g+f^*h)_{ij})}{\det(g_{ij})} < 4$, *or* $f$*: area decreasing map, then*

(i) *The graph of* $f$ *is preserved along MCF; long time existence.*

(ii) *If* $k_1 > 0$ *(locally symmetric), then* $f$ *converges to a constant map.*

## Remark

$$*\Omega = \frac{\sqrt{\det(g_{ij})}}{\sqrt{\det((g+f^*h)_{ij})}} = \frac{1}{\sqrt{\prod_{i=1}^{n}(1+\lambda_i^2)}} > \frac{1}{2} \text{ at } t = 0.$$

# Proof of (i)

$$\frac{\partial}{\partial t}\ln *\Omega = \Delta \ln *\Omega + \mathsf{I} + \mathsf{II}, \quad \text{where}$$

$$\begin{aligned}\mathsf{I} &= \text{second fundamental form terms}\\ &= |A|^2 + \sum_{i,k}\lambda_i^2\left(h_{ik}^{n+i}\right)^2 + 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i}\end{aligned}$$

$$\begin{aligned}\mathsf{II} &= \text{curvature tensor terms}\\ &= \sum_{i,k}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)\left(1+\lambda_k^2\right)}\langle R_1(a_k,a_i)a_k,a_i\rangle - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)\left(1+\lambda_k^2\right)}\langle R_2(a_{n+k},a_{n+i})a_{n+k},a_{n+i}\rangle\right)\\ &= \sum_{i,k\neq i}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)\left(1+\lambda_k^2\right)}K_{N_1}(a_k,a_i) - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)\left(1+\lambda_k^2\right)}K_{N_2}(a_{n+k},a_{n+i})\right)\end{aligned}$$

## Remark

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

$$R_{ijkl} = \langle R(e_k,e_l)e_i,e_j\rangle$$

$$K(e_k,e_i) = \langle R(e_k,e_i)e_k,e_i\rangle, \quad \text{where } \{e_i\} \text{ are orthonormal.}$$

# The "graph" property is preserved by MCF.

- Goal: there exists $\delta > 0$ such that

$$\frac{\partial}{\partial t} \ln *\Omega \geq \Delta \ln *\Omega + \delta |A|^2,$$

  by the maximum principle,
  $\min_{\Sigma_t} \ln *\Omega$ is nondecreasing in $t$, $\Rightarrow *\Omega \geq \min_{\Sigma_{t=0}} *\Omega > 0$.
  Thus $\Sigma_t$ remains the graph of a map $f_t : N_1 \to N_2$ whenever the flow exists.

- At $t = 0$, $\quad \dfrac{\det((g + f^* h)_{ij})}{\det(g_{ij})} = \displaystyle\prod_{i=1}^{n} \left(1 + \lambda_i^2\right) < 4.$

- $N_1$: cpt. Assumption $\Rightarrow \prod_{i=1}^{n} \left(1 + \lambda_i^2\right) \leq 4 - \varepsilon$ on $\Sigma_{t=0}$ for $\varepsilon > 0$.

# Proof of the evolution inequality.

- By continuity and short time existence, the solution remains the graph and $\prod_{i=1}^{n}\left(1+\lambda_i^2\right) \leq 4-\frac{\varepsilon}{2}$ for small $t$.
- In particular, when $i \neq j$, $\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right) \leq 4-\frac{\varepsilon}{2}$. By mean inequality, we have $|\lambda_i \lambda_j| \leq 1-\delta$ for $\delta=\frac{\varepsilon}{8}>0, i \neq j$.
- Thus

$$
\begin{aligned}
\mathrm{I} \geq & \delta|A|^2+(1-\delta) \sum_{i,j,k}\left(h_{jk}^{n+i}\right)^2-2(1-\delta) \sum_{k,i<j}\left|h_{jk}^{n+i} h_{ik}^{n+j}\right| \\
\geq & \delta|A|^2+(1-\delta) \sum_{k,i<j}\left(\left|h_{jk}^{n+i}\right|-\left|h_{ik}^{n+j}\right|\right)^2 \\
\geq & \delta|A|^2 .
\end{aligned}
$$

For curvature tensor terms,

(a) If $k_1 \geq 0, k_2 \leq 0$, we have

$$\mathrm{II} \geq \sum_{i,k\neq i}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_1 - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_2\right) \geq 0.$$

(b) If $k_1 \geq k_2 > 0$, then

$$\begin{aligned}\mathrm{II} &\geq \sum_{i,k\neq i}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_1 - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_2\right)\\ &\geq \sum_{i,k\neq i}\left(\frac{\lambda_i^2-\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\right)k_2 = \sum_{i<k}\left(\frac{\lambda_i^2+\lambda_k^2-2\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\right)k_2\end{aligned}$$

Since $|\lambda_i\lambda_k| < 1, \lambda_i^2+\lambda_k^2-2\lambda_i^2\lambda_k^2 = (\lambda_i-\lambda_k)^2 + 2\lambda_i\lambda_k - 2\lambda_i^2\lambda_k^2 = (\lambda_i-\lambda_k)^2 + 2\lambda_i\lambda_k(1-\lambda_i\lambda_k) \geq 0$.

Hence $\mathrm{II} \geq 0$.

# Long time existence

## Definiton

A *regular point* is a point where the second fundamental form is locally bounded in $2, \alpha$-Hölder norm.

## Theorem (B. White's regularity theorem)

*There is an* $\varepsilon = \varepsilon(n, m, \alpha) > 0$ *such that whenever*

$$\lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0, t_0} d\mu_t < 1 + \varepsilon,$$

*it can concluded that* $(y_0, t_0)$ *is a regular point.*

In our case, we need to define $\rho_{y_0,t_0}$, and "calculate" $\lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t$.

# Isometrically embedding theorem

## Theorem (Nash)

*There are isometric embeddings in* $\mathbb{R}^N, N = \frac{n}{2}(3n+11)$, *of any compact* $n$*-dimensional Riemannian manifold.*

- We isometrically embed $M = N_1 \times N_2$ into $\mathbb{R}^N$.
- The MCF equation $F(x,t)$ in $\mathbb{R}^N$ becomes

$$\frac{\partial}{\partial t} F(x,t) = H = \bar{H} + E,$$

where $H \in TM/T\Sigma_t$: mean curvature of $\Sigma_t$ in $M$, and
$\bar{H} \in T\mathbb{R}^N/T\Sigma_t$: mean curvature of $\Sigma_t$ in $\mathbb{R}^N$.

# $n$-dimensional backward heat kernel (Huisken)

The backward heat kernel $\rho_{y_0,t_0}$ at $(y_0, t_0)$ is

$$\rho_{y_0,t_0} = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \mathrm{e}^{-\frac{|y-y_0|^2}{4(t_0-t)}}.$$

- $\frac{\partial}{\partial t}\rho_{y_0,t_0} = -\Delta\rho_{y_0,t_0} - \rho_{y_0,t_0}\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{F^\perp \cdot \bar{H}}{t_0-t} + \frac{F^\perp \cdot E}{2(t_0-t)}\right)$, where $F \in T\mathbb{R}^N / T\Sigma_t$.
- The monotonicity formula asserts $\lim\limits_{t\to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} \mathrm{d}\mu_t$ exists.
- We hope to show $\lim\limits_{t\to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} \mathrm{d}\mu_t = 1$. However, it is hard to calculate the value directly.

# Parabolic dilation

Consider the parabolic dilation $D_\lambda$ at $(y_0, t_0)$, that is,

$$(y, t) \xrightarrow{D_\lambda} (\lambda(y - y_0), \lambda^2(t - t_0)),$$

and set $s = \lambda^2(t - t_0)$. Denote the corresponding submanifold and volume form after dilation by $\Sigma_s^\lambda$ and $d\mu_s^\lambda$ respectively.

- $E$-almost Brakke flow: view a submanifold as a Radon measure.
- Tangent flow: if the parabolic dilation sequence of $E$-almost Brakke flow $\Sigma_s^\lambda$ converges to a limit $\Sigma_{-1}^\infty$, this limit is called a tangent flow at $(y_0, t_0)$.
- Ilmanen shows the existence of tangent flow.

# Show $(y_0, t_0)$ is a regular point

The quantity

$$\lim_{t\to t_0}\int_{\Sigma_t}\rho_{y_0,t_0}\mathsf{d}\mu_t = \lim_{j\to\infty}\int_{\Sigma_{t_j}}\rho_{y_0,t_0}\mathsf{d}\mu_{t_0+\frac{s_j}{\lambda_j^2}} \overset{(1)}{=} \lim_{j\to\infty}\int_{\Sigma_{s_j}^{\lambda_j}}\rho_{0,0}\mathsf{d}\mu_{s_j}^{\lambda_j}$$

$$\overset{(2)}{=}\int_{\Sigma_{-1}^{\infty}}\rho_{0,0}\mathsf{d}\mu_{-1}^{\infty} = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_{\Sigma_{-1}^{\infty}}\exp\left(-\frac{|F_{-1}^{\infty}|^2}{4}\right)\mathsf{d}\mu_{-1}^{\infty} = 1.$$

(1) $\rho_{0,0} = \frac{1}{\lambda^n}\rho_{y_0,t_0}$, $\mathsf{d}\mu_s^{\lambda} = \lambda^n\mathsf{d}\mu_t$.

(2) We need: $\Sigma_{s_j}^{\lambda_j} \to \Sigma_{-1}^{\infty}$ as Radon measure and $\Sigma_{-1}^{\infty}$ is the graph of a linear function.

# Convergence

- Goal: there exists $c_0 > 0$ which depends on $\varepsilon, k_1, n$ such that
$$\mathsf{II} \geq c_0 \sum_{i=1}^{n} \lambda_i^2 \geq c_0 \ln \left( \prod_{i=1}^{n} \left(1 + \lambda_i^2\right) \right) = -2c_0 \ln *\Omega.$$
Then
$$\frac{\partial}{\partial t} \ln *\Omega \geq \Delta \ln *\Omega - 2c_0 \ln *\Omega.$$
- $*\Omega \to 1$ as $t \to \infty$.
- $|A| \to 0$ as $t \to \infty$.
- $|\mathrm{d}f| \to 0$ as $t \to \infty$.

# Proof of (ii)

Goal: Find $c_0 > 0$ such that $\mathrm{II} \geq c_0 \sum_{i=1}^{n} \lambda_i^2$.

(a) If $k_1 > 0$, and $k_2 \leq 0$, we have

$$\mathrm{II} \geq \sum_{i,k\neq i} \left( \frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)} k_1 - \frac{\lambda_i^2 \lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)} k_2 \right)$$

$$\geq \sum_{i,k\neq i} \frac{\lambda_i^2 k_1}{(1+\lambda_i^2)(1+\lambda_k^2)} \geq \frac{k_1(n-1)}{4} \sum_{i=1}^{n} \lambda_i^2$$

This is because $\dfrac{1}{(1+\lambda_i^2)(1+\lambda_k^2)} \geq \dfrac{1}{\prod_{i=1}^{n}(1+\lambda_i^2)} \geq \dfrac{1}{4}$.

Hence we can take $c_0 = \dfrac{k_1(n-1)}{4}$.

(b) If $k_1 \geq k_2 > 0$, recall

$$\text{II} \geq \sum_{i,k\neq i}\left(\frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_1 - \frac{\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}k_2\right)$$

$$\geq \sum_{i,k\neq i}\left(\frac{\lambda_i^2-\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\right)k_1 = \sum_{i<k}\left(\frac{\lambda_i^2+\lambda_k^2-2\lambda_i^2\lambda_k^2}{(1+\lambda_i^2)(1+\lambda_k^2)}\right)k_1$$

As the proof (i), we have $|\lambda_i\lambda_k| < 1-\frac{\varepsilon}{4}$ for all $t \geq 0$. Thus,

$$\lambda_i^2+\lambda_k^2-2\lambda_i^2\lambda_k^2 = \lambda_i\lambda_k(\lambda_i-\lambda_k)^2 + (1-\lambda_i\lambda_k)(\lambda_i^2+\lambda_k^2) \geq \frac{\varepsilon}{4}(\lambda_i^2+\lambda_k^2)$$

Therefore,

$$\text{II} \geq \frac{\varepsilon k_1}{16}\sum_{i<k}(\lambda_i^2+\lambda_k^2) = \frac{\varepsilon k_1(n-1)}{16}\sum_{i=1}^{n}\lambda_i^2.$$

We can take $c_0 = \dfrac{\varepsilon k_1(n-1)}{16}$.

## Claim: $*\Omega \to 1$ as $t \to \infty$

$$\frac{\partial}{\partial t} \ln *\Omega \geq \Delta \ln *\Omega - 2c_0 \ln *\Omega$$

Consider a function $f(t)$ which depends only on $t$ and satisfies

$$\begin{cases} \frac{\mathsf{d}}{\mathsf{d}t} f(t) = -2c_0 f(t) \\ f(0) = \min\limits_{\Sigma_{t=0}} \ln *\Omega \end{cases} \Rightarrow f(t) = f(0)\mathsf{e}^{-2c_0}t.$$

Then $$\frac{\partial}{\partial t}(\ln *\Omega - f(t)) \geq \Delta(\ln *\Omega - f(t)) - 2c_0(\ln *\Omega - f(t)).$$

By the maximum principle, because $\min\limits_{\Sigma_{t=0}}(\ln *\Omega - f(t)) \geq 0$, we have

$$\min_{\Sigma_{t>0}}(\ln *\Omega - f(t)) \geq 0$$

$$\Rightarrow 0 \geq \ln *\Omega \geq f(0)\mathsf{e}^{-2c_0 t} \text{ on } \Sigma_{t\geq 0} \Rightarrow *\Omega \to 1 \text{ as } t \to \infty.$$

## Claim: $|A| \to 0$ as $t \to \infty$. (locally symmetric case)

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 =& \Delta|A|^2 - 2|\nabla A|^2 + 2\left((\nabla^M_{\partial_k} R)_{\alpha ijk} + (\nabla^M_{\partial_j} R)_{\alpha kik}\right) h^\alpha_{ij} \\ & - 4R_{lijk}h^\alpha_{lk}h^\alpha_{ij} + 8R_{\alpha\beta jk}h^\beta_{ik}h^\alpha_{ij} - 4R_{lkik}h^\alpha_{lj}h^\alpha_{ij} + 2R_{\alpha k\beta k}h^\beta_{ij}h^\alpha_{ij} \\ & + 2\sum_{\alpha,\gamma,i,m}(\sum_k(h^\alpha_{ik}h^\gamma_{mk} - h^\alpha_{mk}h^\gamma_{ik}))^2 + 2\sum_{i,j,m,k}(\sum_\alpha h^\alpha_{ij}h^\alpha_{mk})^2 \\ \le& \Delta|A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2.\end{aligned}$$

The $K_1|A|^4$ term will cause some trouble, so we consider

$$\begin{aligned}&\frac{\partial}{\partial t}\left((*\Omega)^{-2p}|A|^2\right) \\ \le& \Delta\left((*\Omega)^{-2p}|A|^2\right) - (*\Omega)^{-2p}\nabla\left((*\Omega)^{-2p}\right)\cdot\nabla\left((*\Omega)^{-2p}|A|^2\right) \\ & + (*\Omega)^{-2p}\left(|A|^4\left(K_1 - p + 2p(p-1)n\varepsilon_1\right) + K_2|A|^2\right).\end{aligned}$$

(Given $\varepsilon_1 > 0$, there exists $T$ such that $*\Omega > \frac{1}{\sqrt{1+\varepsilon_1}}$ for $t > T$.)

- Choose $\varepsilon_1$ small, and a suitable $p = p(n, \varepsilon_1) \sim \frac{1}{\sqrt{\varepsilon_1}}$ such that the coefficient of the highest order nonlinear term is negative.

- Max. principle, $(\frac{\mathrm{d}f}{\mathrm{d}t} = -K_3 f^2 + K_2 f, f(0) = \max_{t=0}(*\Omega)^{-2p}|A|^2)$, one gets

$$(*\Omega)^{-2p}|A|^2 \leq \frac{K_2}{(\frac{1}{\sqrt{1+\varepsilon_1}})^{1+\sqrt{1+\frac{1}{n\varepsilon_1}}}(\frac{1}{\sqrt{2n\varepsilon_1}} - K_1 - 1)} \to 0 \text{ as } t \to \infty.$$

- It implies that the mean curvature flow of $\Sigma$ converges to a totally geodesic submanifold of $M$.

- Since $*\Omega \to 1$ as $t \to \infty$, we have $|\mathrm{d}f_t| \to 0$ and the limit is a constant map.

# Claim: $|A| \to 0$ as $t \to \infty$. (without locally symmetric)

- $N_1$: compact $\Rightarrow |\nabla^M R|$: bounded. Then

$$\frac{\partial}{\partial t}|A|^2 \leq \Delta |A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2 + K_3.$$

- $|A|^2$: uniform bounded in space and time.
- Show $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}|A|^2 \mathrm{d}\mu_t \leq C$ and $\int_0^\infty \int_{\Sigma_t}|A|^2 \mathrm{d}\mu_t \mathrm{d}t < \infty$. Then

$$\int_{\Sigma_t}|A|^2 \mathrm{d}\mu_t \to 0 \quad \text{as} \quad t \to \infty.$$

- By small $\varepsilon$-regularity theorem (Ilmanen) $\overset{?}{\Rightarrow} \sup_{\Sigma_t}|A|^2 \to 0$ uniformly as $t \to \infty$.

# Area decreasing case

- A map $f : N_1 \to N_2$ is *area-decreasing* if

$$| \wedge^2 \mathrm{d}f|(x) = \sup_{|u\wedge v|=1} |\mathrm{d}f(u) \wedge \mathrm{d}f(v)| < 1 \Leftrightarrow |\lambda_i\lambda_j| < 1 \ \forall \ i \neq j.$$

- Take parallel tensor $S(X,Y) = g(\pi_1(X), \pi_1(Y)) - h(\pi_2(X), \pi_2(Y))$.
- By SVD, we have

$$S = S(E_i, E_j)_{1\le i,j\le n+m} = \begin{pmatrix} B & 0 & D & 0 \\ 0 & I_{(n-r)\cdot(n-r)} & 0 & 0 \\ D & 0 & -B & 0 \\ 0 & 0 & 0 & -I_{(m-r)\cdot(m-r)} \end{pmatrix},$$

$$B_{ij} = S(E_i, E_j) = \frac{1-\lambda_i^2}{1+\lambda_i^2}\delta_{ij}, \quad D_{ij} = S(E_i, E_{n+j}) = -\frac{2\lambda_i}{1+\lambda_i^2}\delta_{ij}$$

# Area decreasing case (continued)

- The sum of two eigenvalue of $S$ is

$$\frac{1-\lambda_i^2}{1+\lambda_i^2}+\frac{1-\lambda_j^2}{1+\lambda_j^2}=\frac{2(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)}$$

- Define $S^{[2]}(w_1 \wedge w_2) = S(w_1) \wedge w_2 + w_1 \wedge S(w_2)$. Then $S^{[2]}$ has eigenvalues $u_i + u_j$. Thus, area-decreasing $\Leftrightarrow$ positivity of $S^{[2]}$.
- By Hamilton's maximum principle to show the positivity of $S^{[2]}$.

# Application I

*Let $N_1, N_2$: cpt. dim $N_1 \geq 2$. Suppose $\exists g, h$ such that $K_{N_1(g)} > 0$ and $K_{N_2(h)} \leq 0$. Then any map from $N_1$ to $N_2$ must be homotopic to a constant map.*

## Proof.

- For $f : (N_1, g) \to (N_2, h)$, its SVD of d$f$ has sing. values $\{\lambda_i\}_{i=1}^n$.
- Since $N_1$: cpt., $\exists$ a positive constant $L$ such that $\lambda_i \lambda_j \leq L$.
- Define $\bar{g} = 2Lg$. The singular values of d$f$ w.r.t. $\bar{g}$ and $h$ will be $\{\bar{\lambda}_i = \frac{\lambda_i}{\sqrt{2L}}\}_{i=1}^n$.
- Therefore, we have $\bar{\lambda}_i \bar{\lambda}_j \leq \frac{1}{2} < 1$ and $K_{N_1(\bar{g})} > 0$. Applying the MCF to the graph of $f$ to get the conclusion.

□

# Application II

2-dilation of a map $f$ between $N_1$ and $N_2$ is said at most $D$ if $f$ maps each 2-dim. submanifold in $N_1$ with volume $V$ to an image with volume at most $DV$.

## Corollary

*Let $(N_1, g), (N_2, h)$: cpt. with $K_{N_1(g)} \geq k_1, K_{N_2(h)} \leq k_2$, $k_1, k_2$: constants. If 2-dilation of $f : (N_1, g) \to (N_2, h)$ is less than $\frac{k_1}{k_2}$, then $f$ is homotopic to a constant map.*

## Proof.

Consider $\bar{g} = k_1 g$ and $\bar{h} = k_2 h$. Then $K_{N_1(\bar{g})} \geq 1, K_{N_2(\bar{h})} \leq 1$, and $f : (N_1, \bar{g}) \to (N_2, \bar{h})$ satisfies $| \wedge^2 \mathrm{d}f| < \frac{k_1}{k_2}\frac{k_2}{k_1} = 1$: area-decreasing. □

# Thank You.