The Isoperimetric Problem in the Heisenberg group $\mathbb{H}^n$

First Taiwan Geometry Symposium, NCTS South

November 20, 2010

Talk outline

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- Relevant terminology, conepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.

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The Euclidean Isoperimetric Problem.

We begin with the following folklore which attributed the Isoperimetric Problem to Queen Dido, founder of the city of Carthage in North Africa.

Figure: Dido, Queen of Carthage. Engraving by Mathäus Merian the Elder 1630.

According to Virgil's saga "Fleeing the vengeance of her brother, Dido (356-260 BC) lands on the coast of North Africa. For the bargain which Dido agrees to with a local potentate is this: she may have that portion of land which she is able to enclose with the hide of a bull. She then cut the hide into a seris of long thin strips and marked out a vast circumference. This area then eventually became the city of Carthage".

- Queen Dido's problem/solution is a variant of what is now known as isoperimetric type problems. In more precise term, Dido's problem is formulated as follows.
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- Over the centuries, the isoperimetric problem (in various forms) has stimulated substantial mathematical research in numerous areas:
- Geometric measure theory: The precise setting for the study of classical questions in the calculus of variations and the proof of existence of an isoperimetric profile. The tools are compactness theorems for *BV* functions. Consequently, a priori solutions are only guaranteed within the class of sets of finite perimeter.

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- PDE: The introduction of dynamic altorihms of volume-constrained curvature flows which provides a way to smoothly deform a given region so that the isoperimetric ratio $P(E)^{\frac{n}{n-1}}/|E|$ cecreases monotonically. If the flow exists for all time, the deformed regions converge, in a suitable sense, to a solution of the isoperimetric problem.
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The Euclidean Isoperimetric Problem.

We recall the classical isoperimetric inequality in the Euclidean space.

Theorem 1

For every Borel set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with finite perimeter $P(\Omega)$,

$$min\{|\Omega|, |\mathbb{R}^n \setminus \Omega|\} \leq C_{iso}(\mathbb{R}^n) P(\Omega)^{\frac{n}{n-1}} , \tag{1}$$

where

$$C_{iso}(\mathbb{R}^n) = \frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}} ,$$

Here, ω_k is the surface measure of the unit sphere S^k in $\mathbb{R}^{k+1}$. Equality holds in (1) if and only if almost everywhere $\Omega = B(x, R)$ (i.e. a ball) for some $x \in \mathbb{R}^n$ and $R > 0$. In the case where $\partial\Omega$ is smooth say C^1 then $P(\Omega)$ coincide with surface measure of $\partial\Omega$. In the non-smooth case $P(\Omega) = Var(\chi_\Omega, \mathbb{R}^n)$ where χ_Ω is the indicator function of Ω and

$$Var(u) = sup\left\{\int_{\mathbb{R}^n} u(x) \sum_{i=1}^{n} \partial_{x_i} G_i \, dx \middle| G_i \in C_o^\infty(\mathbb{R}^n), G_1^2 + \cdots + C_n^2 \leq 1\right\}$$

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Roughly speaking, the *isoperimetric problem* consists in finding the smallest constant $C_{iso}(\mathbb{R}^n)$ and classifying sets Ω such that inequality (1) becomes an equality. This problem is equivalent to the two following formulations:

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The idea of his proof can be outlined in the following three steps. Assume therefore that there is a region $\mathscr{G}$ in the plane such that among all other regions with the same perimeter of $\mathscr{G}$, then $\mathscr{G}$ must be a disc.

Step I: The region $\mathscr{G}$ must be convex. For if not, using reflection, we can construct another region with the same perimeter but enclose a larger area, this contradict our assumption on $\mathscr{G}$.

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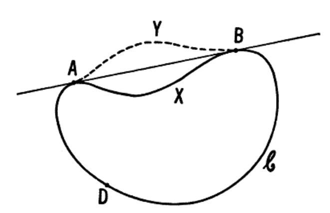

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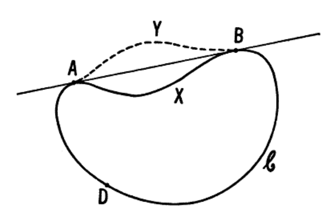

Figure: Steiner's proof, step I.

The Euclidean Isoperimetric Problem.

Step II: Any straight line that divides the perimeter of $\mathscr{G}$ in half must also divide the area of $\mathscr{G}$ in half. Since $\mathscr{G}$ is convex, each half of the bounding curve lies entirely on one side of the line through A and B (see the figures). Suppose the line $\overline{AB}$ does not divide the area of $\mathscr{G}$ in half, reflect the larger area across $\overline{AB}$ to obtain another region having the same perimeter of $\mathscr{G}$ but with a larger area. Again, we obtain a contradiction.

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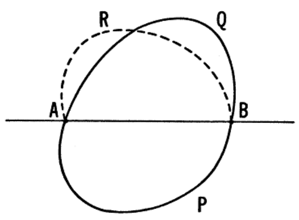

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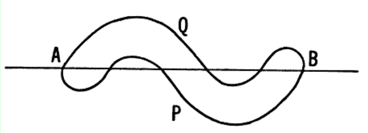

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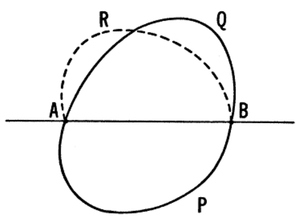

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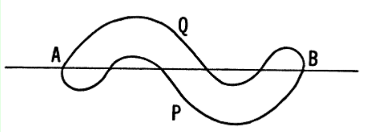

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Step III: Now we concentrate on "half of the figure". Pick any point C on this half curve and join it to A and B to obtain two "lunes" AMC and BNC and a triangle ACB. The angle at C can be either $< \pi/2$, $= \pi/2$ or $> \pi/2$. Imagine that the lunes are made of non-deformable material and they are hinged at C. Now the area of the region is the area of the two lunes plus the area of the triangle. The area of the triangle is computed by the base $\overline{AC}$ and height $\overline{BD}$. By adjusting the angle at C by moving the lunes, we see that the largest height is obtained when $B = D$, that is the angle at C is $\pi/2$.

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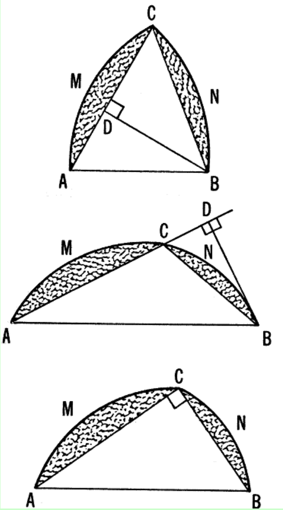

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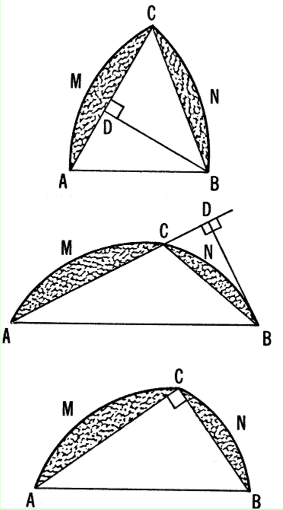

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Since Steiner's proof there are many many proofs for the isoperimetric inequality. We present a few for the $\mathbb{R}^2$ case.

- Proof by complex function theory. Let $z = x + iy$ and $dA = dx \wedge dy = \frac{1}{2} i dz \wedge d\bar{z}$. Using the fact that winding number of $\partial\Omega$ is one, Green and Fubini's theorem we find

$$4\pi A = \int_\Omega 2\pi i \, dz \wedge d\bar{z} = \int_\Omega \int_{\partial\Omega} \frac{d\xi}{\xi - z} dz \wedge d\bar{z} = \int_{\partial\Omega} \int_{\partial\Omega} \frac{\bar{\xi} - \bar{z}}{\xi - z} dz \, d\xi \leq L^2 \, .$$

- The case of equality is easy to analyze in the above. The interplay between geometric extremal problems (e.g. isoperimetric problem) and sharp analytic inequalities is witnessed in the following analytic proof of the planar isoperimetric inequality. First, let's recall Wirtinger's inequality. If f is in the Sobolev space $W^{1,2}([0, 2\pi])$ satisfying $\int_0^{2\pi} f(t) dt = 0$ then

$$\int_0^{2\pi} |f(t)|^2 dt \leq \int_0^{2\pi} |f'(t)|^2 dt \, , \tag{2}$$

with equality holds only when $f(t) = A cos(t) + B sin(t)$. The proof of Wirtinger's inequality is an easy exercise in Fourier series.

Let ds denote the element of arc length and assume that $\partial\Omega$ is a Lipschitz curve which is the boundary of a domain $\Omega \subset \mathbb{R}^2$. Denote by $x = (x_1, x_2)$ the position vector. By translating the region Ω which preserves area and perimeter, we may assume that $\int_{\partial\Omega} x ds = 0$. The divergence theorem and Wirtinger's inequality applied to x_1, x_2 then gives

$$2A = \int_\Omega div(x)\, dA = \int_{\partial\Omega} < x, \vec{n} > ds \le \int_{\partial\Omega} |x| ds \le \sqrt{L}\left(\int_{\partial\Omega} |x|^2 ds\right)^{\frac{1}{2}}$$

$$\le \sqrt{L}\left[\left(\frac{L}{2\pi}\right)^2 \int_{\partial\Omega} \left|\frac{dx}{ds}\right|^2 ds\right]^{\frac{1}{2}} \le \frac{L^2}{2\pi}\ .$$

Equality holds if and only if Ω is a disc.

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The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- The Heisenberg group $\mathbb{H}^n$ is a Lie group on $\mathbb{R}^{2n+1}$ with the following group law:

$$(x, y, t) \circ (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x)\right).$$

  where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, $t \in \mathbb{R}$ and the dot product is the standard dot product on Euclidean spaces.

- Associated to this group law we work with the following standard left-invariant vector fields: $(i = 1..n)$

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}, \quad T = [X_i, Y_i] = \frac{\partial}{\partial t},$$

  together with an inner product $<,>$ with respect to which these $2n+1$ vector fields form an orthonormal system. $\mathbb{H}^n$ equiped with $<,>$ is then a Riemannian manifold.

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together with an inner product $<,>$ with respect to which these $2n+1$ vector fields form an orthonormal system. $\mathbb{H}^n$ equiped with $<,>$ is then a Riemannian manifold.

- The Lebesgue measure on $\mathbb{R}^{2n+1}$ is both left and right translation measure and therefore a Haar measure on $\mathbb{H}^n$. For sets $E \subset \mathbb{H}^n$, the volume of E is the Lebesgue measure of E and will be denoted by $|E|$.

The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- The Heisenberg group $\mathbb{H}^n$ is a Lie group on $\mathbb{R}^{2n+1}$ with the following group law:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x)) .$$

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- Given an oriented C^2 embedded hypersurface $\mathscr{S} \subset \mathbb{H}^n$ (and after an orientation is chosen) we let N be the Riemannian unit normal to $\mathscr{S}$ and we write $N = \sum_{i=1}^n (p_i X_i + q_i Y_i) + \omega T$
- The projection of N onto the horizontal plane $span\{X_i, Y_i \mid i = 1, .., n\}$ at each point $g \in \mathscr{S}$ is called the *horizontal normal* and is denoted by $N_H = \sum_{i=1}^n p_i X_i + q_i Y_i$.

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- The set

$$\Sigma_{\mathscr{S}} \stackrel{def}{=} \{g \in \mathscr{S} \mid N_H(g) = 0\}$$

is called the *characteristic set* (*singular set* by some authors) of the hypersurface $\mathscr{S}$. For our purpose, it suffices to know that for any C^2 hypersurface $\mathscr{S}$ we have $\sigma(\Sigma_{\mathscr{S}}) = 0$ where $d\sigma$ is the Riemannian volume on $\mathscr{S}$.

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...relevant concept... Recall H-Gauss map $\nu_H = \sum_{i=1}^{n} \overline{p}_i X_i + \overline{q}_i Y_i$.

An important geometric quantity that sterms from ν_H is the so called Horizontal mean curvature (H-mean curvature hereafter) $\mathscr{H}$.

- For $g_o \in \mathscr{S} \setminus \Sigma_{\mathscr{S}}$ define

$$\mathscr{H}(g_0) = div_H(\nu_H) = \sum_{i=1}^{n} X_i \overline{p}_i + Y_i \overline{q}_i \,,$$

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- If $E \subset \mathscr{S}$ is a smooth portion of $\mathscr{S}$, the *H-surface measure* (or *H-area*) of E is the quantity

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For $u \in L^1_{loc}(\mathbb{H}^n, dg)$ where dg is the Lebesgue measure on $\mathbb{H}^n$ we define the Horizontal variation of u by

$$Var_H(u) = sup\left\{\int_{\mathbb{H}^n} u \sum_{i=1}^{n} X_i\xi_i + Y_i\eta_i \, dg \,\middle|\, \sum_{i=1}^{n} \xi_i^2 + \eta_i^2 \leq 1 \;, \xi_i, \eta_i \in C_o^\infty(\mathbb{H}^n), i = 1..n\right\}$$

If $u \in L^1(\mathbb{H}^n, dg)$ is such that $Var_H(u) < \infty$ we say that u is of bounded H-variation. For any sets $E \subset \mathbb{H}^n$, we define the horizontal perimeter of E by

$$P_H(E) = Var_H(\chi_E) \qquad \text{where } \chi_E \text{ is the indicator function of } E\,.$$

If $P_H(E) < \infty$ we say that E is of finite H-perimeter. We note here that if E is smooth say (∂E is) C^1, then $P_H(E) = \sigma_H(\partial E)$.

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Homogeneous structure on $\mathbb{H}^n$.

- For $\lambda \neq 0$, define a family of dilations on $\mathbb{H}^n$ to be a function $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Here $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

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- For a fixed $g_o \in \mathbb{H}^n$, the left translation is the map $\tau_{g_o} : \mathbb{H}^n \to \mathbb{H}^n$ given by $\tau_{g_o}(g) = g_o \circ g$ where $\circ$ is the group law on $\mathbb{H}^n$.

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- We note here that both the Lebesgue measure and the H-perimeter behave nicely with respect to left translation and dilation: For any measurable set $E \subset \mathbb{H}^n$:

$$\text{for any } g_o \in \mathbb{H}^n \,,\ |\tau_{g_o}(E)| = |E| \,, \qquad P_H(\tau_{g_o}(E)) = P_H(E)$$
$$\text{for any } \lambda > 0 \,,\ |\delta_\lambda(E)| = \lambda^Q |E| \,, \qquad P_H(\delta_\lambda(E)) = \lambda^{Q-1} P_H(E) \,.$$

In the above, the number $Q = 2n + 2$ is called the homogenous dimension of $\mathbb{H}^n$.

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Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in R$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2)^2 + 16t^2)^{\frac{1}{4}}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_\lambda(g)||_{\mathbb{H}^n} = |\lambda|||g||_{\mathbb{H}^n}$.
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- There is however another distance from the Sub-Riemannian point of view. For any piecewise C^1 curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that γ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in Span\{X_1, ..., Y_n\}$.

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- If d denotes either d_{CC} or $d_{\mathbb{H}^n}$, then it is easy to verify that d is translation invariant and homogeneous of degree one, i.e.

$$d(g'' \circ g, g'' \circ g') = d(g, g') \quad \text{and } d(\delta_\lambda(g), \delta_\lambda(g')) = |\lambda|\, d(g, g')\ .$$

- The two distances are comparable in the sense that there exist constants $c = c(\mathbb{H}^n), C = C(\mathbb{H}^n)$ such that for all $g, g' \in \mathbb{H}^n$ we have $c\, d_{\mathbb{H}^n}(g, g') \leq d_{CC}(g, g') \leq C d_{\mathbb{H}^n}(g, g')$. However, the shape of balls with respect to these two distances are quite different as we will see.
- Now we can consider the Isoperimetric problem in the Heisenberg group.

- If d denotes either d_{CC} or $d_{\mathbb{H}^n}$, then it is easy to verify that d is translation invariant and homogeneous of degree one, i.e.

$$d(g'' \circ g, g'' \circ g') = d(g, g') \quad \text{and } d(\delta_\lambda(g), \delta_\lambda(g')) = |\lambda|\, d(g, g')\ .$$

- The two distances are comparable in the sense that there exist constants $c = c(\mathbb{H}^n), C = C(\mathbb{H}^n)$ such that for all $g, g' \in \mathbb{H}^n$ we have $c\, d_{\mathbb{H}^n}(g, g') \le d_{CC}(g, g') \le C d_{\mathbb{H}^n}(g, g')$. However, the shape of balls with respect to these two distances are quite different as we will see.
- Now we can consider the Isoperimetric problem in the Heisenberg group.

- The Isoperimetric problem began with Pansu's work in 1982. Using an idea of Croke, Pansu established the following Isoperimetric inequality in $\mathbb{H}^1$: There exist a constant $C > 0$ so that $|\Omega|^{\frac{3}{4}} \leq CP_H(\Omega)$ for any bounded open set $\Omega \subset \mathbb{H}^1$ with C^1 boundary. To state Pansu's conjecture, we introduce the following

Definition 2

The isoperimetric constant of the Heisenberg group $\mathbb{H}^n$ is the best constant for which the isoperimetric inequality $min\{|\Omega|^{\frac{Q-1}{Q}}, |\mathbb{H}^1 \setminus \Omega|^{\frac{Q-1}{Q}}\} \leq C_{iso}(\mathbb{H}^n) P_H(\Omega)$, that is

$$C_{iso}(\mathbb{H}^n) = sup\left\{ \frac{min\{|\Omega|^{\frac{Q-1}{Q}}, |\mathbb{H}^n \setminus \Omega|^{\frac{Q-1}{Q}}\}}{P_H(\Omega)} \,\middle|\, 0 < P_H(\Omega) < \infty \right\},$$

An isoperimetric profile of parameter $V > 0$ for $\mathbb{H}^n$ consists of a family of bounded sets $\mathscr{B}_V$ with $|\mathscr{B}_V| = V$ and $|\mathscr{B}_V|^{\frac{Q-1}{Q}} = C_{iso}(\mathbb{H}^n)\, P_H(\mathscr{B}_V)$. Since $|\cdot|$ and $P_H(\cdot)$ are invariant under left translation and scaling property hold for them, the class of isoperimetric profile is preserved under left translation and group dilation.

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Pansu's Isoperimetric inequality and conjecture.

Pansu conjectured in 1984 that the isoperimetric profiles on $\mathbb{H}^1$ is obtained by revolving around the t-axis the geodesics (with respect to the Carnot-Caratheodory metric) joining the points $(0,0,\pi R^2/8)$ and $(0,0,-\pi R^2/8)$. For the moment, $R>0$ is just a parameter. These geodesic can be easily obtained as: $\gamma:[-\pi,\pi]\to\mathbb{H}^1$

$$\gamma(s)=\left(\frac{R}{2}(cos(s)+1),\frac{R}{2}sin(s),\frac{R^2}{8}(sin(s)+s)\right)$$

With the explicit form of the geodesics, it is an easy excercise to compute and to obtain $C_{iso}(\mathbb{H}^1)=\frac{3^{\frac{3}{4}}}{4\sqrt{\pi}}$

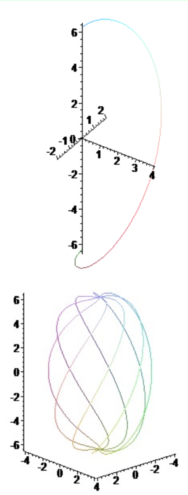

Figure: An illustration of Pansu's conjecture with $R=4$

We have to stress that until today, Pansu's conjecture has not completely been solved in the greatest generality, that is, if we consider the largest admissible sets: $\Omega \subset \mathbb{H}^n$ for which $P_H(\Omega) < \infty$ without any regularity assumption on $\partial\Omega$. In the remaining time, we survey results and some ideas used in the pursue of Pansu's conjecture. The first question that one may have is why the gauge balls or the CC-metric balls are not solution to the isoperimetric problem (i.e., not the isoperimetric profiles).

- The gauge balls are smooth and one can compute its H-mean curvature and it is not constant. Later on, we see that a necessary condition for a smooth set to be a solution to the isoperimetric problem, its H-mean curvature must be constant.

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The isoperimetric profile from Pansu's conjecture can be described more explicitly as follows.

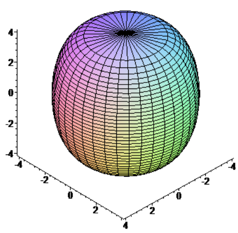
$$\partial\mathscr{B}_R(0) = \{(x,y,t) \in \mathbb{R}^{2n+1} \mid t = \pm u(x,y) = u(|z|)\}$$

where

$$u(x,y) = u(|z|) = \frac{|z|\sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} arcsin\left(\frac{|z|}{R}\right) + \frac{\pi R^2}{8}\ . \tag{3}$$

In the above $|z|^2 = |x|^2 + |y|^2$.

The gauge balls, CC-balls and the isoperimetric profile from Pansu's conjecture and is called now a days the Heisenberg bubbles. Note that CC-balls are not smooth.



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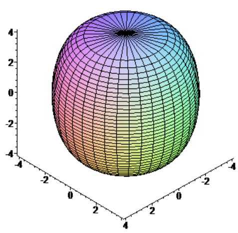

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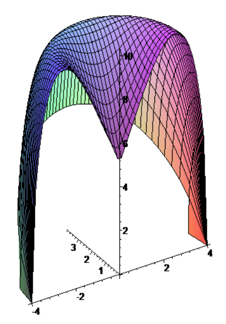

Figure: A quarter of the CC-ball with $R = 4$

Balls are not solution to the isoperimetric problem.

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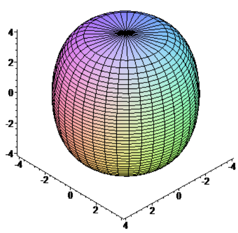

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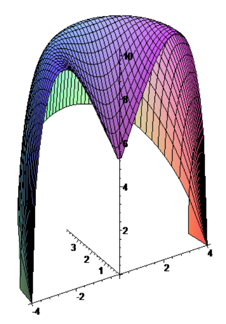

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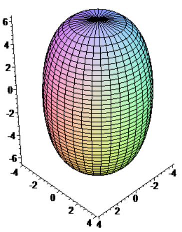

Figure: The isoperimetric bubble, $R = 4$

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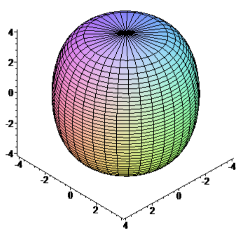

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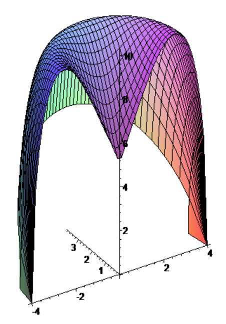

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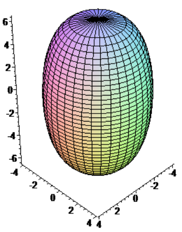

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We turn to the first positive result in this direction: The existence result.

Theorem 3 (Leonardi-Rigot, 2003)

For any $V > 0$*, there exists a bounded set* $\Omega \subset \mathbb{H}^n$ *with* $P_H(\Omega) < \infty$, $|\Omega| = V$ *and* $|\Omega|^{\frac{Q-1}{Q}} = C_{iso}(\mathbb{H}^n) P_H(\Omega)$.

Remark 4

Leonardi and Rigot's result continue to hold in all Carnot-groups where the Heisenberg groups $\mathbb{H}^n$, $n \geq 1$ *are the simplest such examples of step 2.*

The proof of Theorem 3 consists of the following ideas.

- An important ingredient due to Garofalo-Nhieu (1996) states that if $\{\Omega_n\}$ is a sequence of measurable sets in $\mathbb{H}^n$ (in fact, in more greater generality) with $sup\{P_H(\Omega_n)\} < \infty$ then there is a subsequence still denoted by $\{\Omega_n\}$ and a measurable set Ω with $P_H(\Omega) < \infty$ and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{H}^n)$.
- They considered a sequence $\{\Omega_n\}$ such that $|\Omega_n| = 1$ and that

$$\frac{|\Omega_n|^{\frac{Q-1}{Q}}}{P_H(\Omega_n)} = \frac{1}{P_H(\Omega_n)} \to C_{iso}(\mathbb{H}^n)$$

with $P_H(\Omega_n) \leq C_{iso}(\mathbb{H}^n)^{-1}(1+1/n)$. Using the above theorem, they obtain a subsequence $\{\Omega_n\}$ converging in $L^1_{loc}(\mathbb{H}^n)$ to Ω with $|\Omega| = 1$.

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- To show that that Ω is bounded, they established the following lemma that prevents the possibility that the sets Ω_n become very thin, spread out and in the limit lose volume at infinity. That is, for each Ω_n, a fixed amount of volume must lie within a ball of radius one.

Lemma 1 ("Concentration-Compactness")

Let A be a set with $0 < |A|, P_H(A) < \infty$. If $m \in (0, |B(0,1)|/2)$ is such that $|A \cap B(g,1)| < m$ for all $g \in \mathbb{H}^n$ then there is a constant $c > 0$ so that

$$c\left(\frac{|A|}{P_H(A)}\right)^Q \leq m\,.$$

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- The lemma plus a bit of work show that the limiting set Ω is essentially bounded. Finally, They established that $\partial\Omega$ satisfies (1) Ahlfors regularity condition and (2) interior and exterior corkscrew condition. Essential boundedness together with (1) and (2) imply that Ω is bounded.

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$$\frac{d}{d\rho}\left(\frac{\rho^{2n-1} f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}}\right) = \lambda_n \rho^{2n-1}\ , \quad f'(0) = 0\ .$$

Smooth Cylindrical Case and some properties of the Isoperimetric profile: Leonardi-Masnou, Ritoré-Rosales.

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By solving the above equation, they obtain the formula (3) i.e., the Heisenberg bubbles. They also establish some properties of such sets which can be summarized as follows

Theorem 5 (Leonardi-Masnou, 2005)

There exists, up to dilations and left translations, sets Ω given by (3) are critical points of the isoperimetric problem in $\mathbb{H}^n$. Furthermore, the H-mean curvature of $\partial\Omega$ is constant and $\partial\Omega$ are foliated by the geodesics joining the North and South poles of Ω in Pansu's conjecture.

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Definition 6

Let $\Omega \subset \mathbb{H}^n$ be a C^2 bounded set such that $\partial\Omega$ is an embedded surface with $P_H(\Omega) = \sigma_H(\partial\Omega) < \infty$ and U a C^1 vector field with compact support on $\mathbb{H}^n$. For small ϵ denote by $S_\epsilon = \{exp(\epsilon U_p) \mid p \in \partial\Omega\}$ the variation of $\partial\Omega$ induced by U. We let Ω_ϵ the region enclosed by S_ϵ and define $P(\epsilon) = \sigma_H(S_\epsilon)$, $V(\epsilon) = |\Omega_\epsilon|$.

It is well known that

$$V'(0) = \int_\Omega div(U) dg = -\int_{\partial\Omega} < U, N > d\sigma \tag{4}$$

where N is the Riemannian unit normal pointing into Ω, $d\sigma$ is the Riemannian volume on $\partial\Omega$.

Some properties of the isoperimetric profile.

Theorem 7 (Ritoré-Rosales, 2006 "First variation formula")

If Ω is such that $V'(0) = 0$ and the H-mean curvature $\mathcal{H}$ of $\partial\Omega$ is in $L^1_{loc}(\partial\Omega, d\sigma)$ then

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Since a C^2 solution Ω to the isoperimetric problem must be a critical point of the H-perimeter functional (i.e. $P'(0) = 0$) that preserves volume (i.e. $V'(0) = 0$ for all C^1 vector fields U with compact support in $\mathbb{H}^n$) we have (in view of Theorem 11 and (4)

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(i) $|E \cap \mathbb{H}^n_+| = |E \cap \mathbb{H}^n_-|$;

(ii) there exist $R > 0$, and functions $u, v : \overline{B}(0,R) \to [0,\infty)$, with $u, v \in C^2(B(0,R)) \cap C(\overline{B}(0,R))$, $u = v = 0$ on $\partial B(0,R)$, and such that

$$\partial E \cap \mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n_+ \mid |z| < R,\ t = u(z)\},$$

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In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou's result by relaxing some symmetry conditions. We also show that Pansu's conjecture is not only a critical point but indeed a minimizer of the H-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $\mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n \mid t > 0\}$, $\mathbb{H}^n_- = \{(z,t) \in \mathbb{H}^n \mid t < 0\}$, and consider the collection $\mathscr{E} = \{E \subset \mathbb{H}^n \mid E \quad \text{satisfies } (i) - (iii)\}$, where

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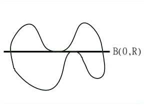

Figure: $E \in \mathscr{E}$

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Partial Symmetry Case: Danielli-Garofalo-Nhieu.

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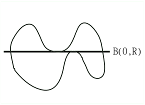

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Remark 9

We note explicitly that condition

$$(iii)\ \{z \in B(0,R) \mid u(z) = 0\} \cap \{z \in B(0,R) \mid v(z) = 0\} = \varnothing$$

serves to guarantee that every $E \in \mathcal{E}$ *is a piecewise* C^2 *domain in* $\mathbb{H}^n$ *(with possible discontinuities in the derivatives only on that part of* E *which intersects the hyperplane* $t = 0$*). We also stress that the upper and lower portions of a set* $E \in \mathcal{E}$ *can be described by possibly different* C^2 *graphs, and that, besides* C^2 *smoothness, and the fact that their common domain is a ball, no additional assumption is made on the functions* u *and* v*. For instance, we do not require a priori that* u *and/or* v *are spherically symmetric. Here is our main result.*

Partial Symmetry: (3) $u(x,y) = u(|z|) = \frac{|z|\sqrt{R^2-|z|^2}}{4} - \frac{R^2}{4} arcsin\left(\frac{|z|}{R}\right) + \frac{\pi R^2}{8}$.

Theorem 10 (Danielli-Garofalo-Nhieu, 2008)

Let $V>0$, and define the number $R>0$ by

$$R = \left(\frac{(Q-2)\Gamma\left(\frac{Q+2}{2}\right)\Gamma\left(\frac{Q-2}{2}\right)}{\pi^{\frac{Q-1}{2}}\Gamma\left(\frac{Q+1}{2}\right)} \right)^{1/Q} V^{1/Q}.$$

Given such R, then the variational problem $\min_{E\in\mathscr{E},|E|=V} P_H(E;\mathbb{H}^n)$ *has a unique solution* $E_R \in \mathscr{E}$, *where* ∂E_R *is described by the graph* $t = \pm u(x,y)$ *i.e.* (3) *The boundary of* E_R *is only of class* C^2, *but not of class* C^3, *near its two singular points* $\left(0, \pm\frac{\pi R^2}{8}\right)$, *it is* C^∞ *away from them, and* ∂E_R *has positive constant H-mean curvature and isoperimetric constant given respectively by*

$$\mathscr{H} = \frac{Q-2}{R}, \qquad C(\mathbb{H}^n) = \frac{(Q-1)\Gamma\left(\frac{Q}{2}\right)^{\frac{2}{Q}}}{Q^{\frac{Q-1}{Q}}(Q-2)\Gamma\left(\frac{Q+1}{2}\right)^{\frac{1}{Q}}\pi^{\frac{Q-1}{2Q}}}.$$

Our proof is based on the following ideas.

- Under the assumption of sets $E \in \mathscr{E}$, the variational problem $\min\limits_{E \in \mathscr{E}, |E| = V} P_H(E; \mathbb{H}^n)$ is equivalent to minimizing the unconstrained functional with a Lagrange multiplier λ to be properly chosen:

$$\mathscr{F}[u] = \int_{supp(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} dz \quad z = (x,y) \in \mathbb{R}^{2n} \qquad (5)$$

- We easily recognize that the Euler-Lagrange equation of (5) is

$$div_z \left[\frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + < \nabla_z u, z^\perp >}} \right] = \lambda \quad z^\perp = (-y, x) \in \mathbb{R}^{2n}. \qquad (6)$$

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The best smooth C^2 result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

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We now describe the proof of Ritoré-Rosales' theorem. In what follows, we let Ω to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial\Omega$ is C^2, compact, orientable.

I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial\Omega}$ contains a C^1 curve C, then the rules of Legendrian foliation of $\partial\Omega$ meet C orthogonally.

II Improved regularity of singular curves: If $\Sigma_{\partial\Omega}$ contains a C^1 curve C, then infact C is C^2. This result leads to a crucial property of the singular curves.

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IV By step III, any compact C^2 solution to the isoperimetric problem cannot have a nontrival curve in its singular set since such curve would be a geodesic which will leave any bounded domain in a finite time. Thus $\Sigma_{\mathscr{S}}$ consists of isolated points.

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IV By step III, any compact C^2 solution to the isoperimetric problem cannot have a nontrival curve in its singular set since such curve would be a geodesic which will leave any bounded domain in a finite time. Thus $\Sigma_{\mathscr{S}}$ consists of isolated points.

V If $\mathscr{S}$ is any C^2, connected, complete oriented and immerse surface in $\mathbb{H}^1$ with constant H-mean curvature and if $\Sigma_{\mathscr{S}}$ contains an isolated point, then $\mathscr{S}$ is congruent to the boundary of a bubble set.

The C^2 solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

We now describe the proof of Ritoré-Rosales' theorem. In what follows, we let Ω to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial\Omega$ is C^2, compact, orientable.

- I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial\Omega}$ contains a C^1 curve C, then the rules of Legendrian foliation of $\partial\Omega$ meet C orthogonally.
- II Improved regularity of singular curves: If $\Sigma_{\partial\Omega}$ contains a C^1 curve C, then infact C is C^2. This result leads to a crucial property of the singular curves.
- III If $\partial\Omega$ is complete surface with non-vanishing H-mean curvature, then any connected curve in $\Sigma_{\mathscr{S}}$ is a geodesic.
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We now describe two results in the directions of finding the isoperimetric profile among non-smooth sets.

Definition 13

A set $E \subset \mathbb{H}^n$ is axially symmetric if $(z,t) \in E$ implies $(\xi, t) \in E$ for all $\xi \in \mathbb{R}^{2n}$ such that $|\xi| = |z|$. Let $\mathscr{A}$ denote the collections of axially symmetric sets in $\mathbb{H}^n$.

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The isoperimetric profiles in $\mathbb{H}^n$ up to a vertical translation, a dilation and a $2n+1$-Lebesgue measure negligible set, are still given by the bubble set described by (3) if restricted to the class $\mathscr{A}$

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The non-smooth cases: Monti, Monti-Rickly.

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The most recent result up to now for the isoperimetric problem in $\mathbb{H}^n$ is again due to Ritoré. He generalized the results of Danielli-Garofalo-Nhieu by removing that the upper and lower part of $\partial\Omega$ to be graphs. To be precise, he proved the following theorem. Let $D_r = \{(z,0) \,|\, |z| < r\} \subset \mathbb{R}^{2n}$ be the Euclidean ball centered at 0 with radius r. $C_r = \{(z,t) \,|\, z \in D_r, t \in \mathbb{R}\}$. We also denote the region enclosed by the Heisenberg bubbles in $\mathbb{H}^n$ by $\mathscr{B}_r$.

Theorem 17

Let $\Sigma \subset \mathbb{H}^n$ be such that $P_H(\Sigma) < \infty$ and $D_r \subset \Sigma \subset C_r$ for some $r > 0$. Then $P_H(\Sigma) \geq P_H(\mathscr{B}_R)$. Equality holds if and only if $\Sigma = \mathscr{B}_R$.

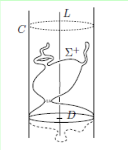

Figure: Conditions for the sets Σ.

A rough sketch of the proof is the following. On the cylinder C_r, two foliations by vertically translating the upper and lower boundary of $\mathscr{B}_r$ are constructed. Using these foliations, he proved that the bubble sets $\partial\mathscr{B}_\lambda$ minimize the functional "H-perimeter - $n\lambda$ volume" in the class of sets E mentioned above. Then minimize over all bubble sets $\mathscr{B}_\mu$ the functional "H-perimeter - $n\mu$(volume - $|\Sigma|$) to obtained the desired result. Another important result due to Monti-Vittone is used to deal with the issue of regularity: a set in $\mathbb{H}^n$ with locally finite H-perimeter with continuous horizontal unit normal has H-regular boundary.

Thank you for your attention!