Advanced Calculus (I)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

1.1 Ordered Field Axioms

Postulate 1: [Field Axioms]
There are functions $+$ and $\cdot$, defined on $\mathbf{R} := \mathbf{R} \times \mathbf{R}$, which satisfy the following properties for every $a, b, c \in \mathbf{R}$

Closure Properties.
$a + b$ and $a \cdot b$ belong to $\mathbf{R}$

Associative Properties.
$a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Commutative Properties.
$a + b = b + a$ and $a \cdot b = b \cdot a$

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Distributive Law.

$a \cdot (b + c) = a \cdot b + a \cdot c$

Existence of the Additive Identity.

There is a unique element $0 \in \mathbf{R}$ such that $0 + a = a$ for all $a \in \mathbf{R}$

Existence of the Multiplicative Identity

There is a unique element $1 \in \mathbf{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbf{R}$

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Existence of Additive Inverse.

For every $x \in \mathbf{R}$ there is a unique element $-x \in \mathbf{R}$ such that

$$x + (-x) = 0$$

Existence of Multiplicative Inverse.

For every $x \in \mathbf{R} \setminus \{0\}$ there is a unique element $x^{-1} \in \mathbf{R}$ such that

$$x \cdot (x^{-1}) = 1$$

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Postulate 2: [Order Axioms]

There is a relation $<$ on $\mathbf{R} \times \mathbf{R}$ that has the following properties:

Trichotomy Properties.

Given $a, b \in \mathbf{R}$, one and only one of the following statements holds:

$$a < b, \quad b < a, \quad or \quad a = b$$

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Additive Properties.

$$a < b \quad c \in \mathbf{R} \quad imply \quad a + c < b + c$$

Multiplicative Properties.

$$a < b \quad and \quad c > 0 \quad imply \quad ac < bc$$

and

$$a < b \quad and \quad c < 0 \quad imply \quad bc < ac$$

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1.1 Remark

We assume that **N** and **Z** satisfy the following properties.

(i)Given $n \in \mathbf{Z}$, one and only one of the following statements holds:

$$n \in \mathbf{N} \quad -n \in \mathbf{N} \quad or \quad n = 0$$

(ii) If $n \in \mathbf{N}$, then $n + 1 \in \mathbf{N}$ and $n \geq 1$

(iii)If $n \in \mathbf{N}$, and $n \neq 1$, then $n - 1 \in \mathbf{N}$

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Theorem

The absolute value satisfies the following three properties.

(i)[Positive Definite]

For all $a \in \mathbf{R}$, $|a| \geq 0$ *with* $|a| = 0$ *if and only if* $a = 0$

(ii)[Symmertic]

For all $a, b \in \mathbf{R}$, $|a - b| = |ba|$

(iii)[Triangle Inequalities]

For all $a, b \in \mathbf{R}$

$$|a+b| \leq |a| + |b|, \quad |a-b| \geq |a| - |b|, \quad ||a| - |b|| \leq |a-b|$$

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Theorem

Let $x, y, a \in \mathbf{R}$

(i)$x < y + \epsilon$ *for all* $\epsilon > 0$ *if and only if* $x \leq y$

(ii)$x > y - \epsilon$ *for all* $\epsilon > 0$ *if and only if* $x \geq y$

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