Advanced Calculus (I)

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1.3 The Completeness Axiom

Definition

Let $E \subset \mathbf{R}$ be nonempty

(i) The set E is said to be *bounded above* if and only if there is an $M \in \mathbf{R}$ such that $a \leq M$ for all $a \in E$

(ii) A number M is called an *upper bound* of the set E if and only if $a \leq M$ for all $a \in E$

(iii) A number s is called *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E *has a supermum* s and shall write s=sup E.)

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If a set has a supremum, then it has only one supremum.

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Proof:

Let s_1 and s_2 be suprema of the same set E. Then both s_1 and s_2 are upper bounds of E, whence by Definition 1.16(iii), $s_1 \leq s_2$ and $s_2 \leq s_1$. We conclude by the

TrichotomyProperty that $s_1 = s_2$ □

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Theorem (Approximation Property For Suprema)

If E has a supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that

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Postulate 4:[Completeness Axiom]

If E is a nonempty subset of $\mathbf{R}$ that is bounded above, then E has a (finite) supermum.

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Theorem (Archimedean Principle)

Given positive real numbers a and b, there is an integer $n \in N$ *such that* $b < na$.

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Theorem (Density of Rationals)

If $a, b \in \mathbf{R}$ satisfy $a < b$, then there is a $q \in Q$ such that $a < q < b$.

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Theorem (Monotone Property)

Suppose that $A \subseteq B$ are nonempty subsets of $\mathbf{R}$

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