Advanced Calculus (II)

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Ch11.6: Inverse Function Theorem

Notation:
$f : \mathbf{R}^n \to \mathbf{R}^n$,
The Jacobian of f is

$$\Delta_f(a) := \det(Df(a)).$$

Lemma (11.38)

Let V be open in $\mathbf{R}^n$, $f : V \to \mathbf{R}^n$, $\mathbf{a} \in \mathbf{V}$, and $r > 0$ be so small that $\overline{B_r(\mathbf{a})} \subset V$. Suppose that f is continuous and 1-1 on $\overline{B_r(\mathbf{a})}$, and its first-order partial derivatives exist at every point in $B_r(\mathbf{a})$. If $\Delta_f \neq 0$ on $B_r(\mathbf{a})$, then there is a $\rho > 0$ such that $B_\rho(f(\mathbf{a})) \subset f(B_r(\mathbf{a}))$.

Theorem (11.39)

Let V be open and nonempty in $\mathbf{R}^n$, and $f : V \to \mathbf{R}^n$ be continuous. If f is 1-1 and has first-order partial derivatives on V, and if $\Delta_f \neq 0$ on V, then f^{-1} is continuous on $f(V)$.

Lemma (11.40)

Let V be open in $\mathbf{R}^n$ and $f : V \to \mathbf{R}^n$ be $\mathcal{C}^1$ on V. If $\Delta_f(\mathbf{a}) \neq 0$ for some $\mathbf{a} \in V$, then there is an $r > 0$ such that $B_r(\mathbf{a}) \subset V$, f is 1-1 on $B_r(\mathbf{a})$, $\Delta_f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in B_r(\mathbf{a})$, and

$$\det \left[\frac{\partial f_i}{\partial x_j}(c_j) \right]_{n \times n} \neq 0$$

for all $\mathbf{c}_1, \ldots, \mathbf{c}_n \in B_r(\mathbf{a})$.

Theorem (11.41 Inverse Function Theorem)

Let V be open in $\mathbf{R}^n$ and $f : V \to \mathbf{R}^n$ be $\mathcal{C}^1$ on V. If $\Delta_f(\mathbf{a}) \neq 0$ for some $\mathbf{a} \in V$, then there exists an open set W containing $\mathbf{a}$ such that

(i) f is 1-1 on W,

(ii) f^{-1} is $\mathcal{C}^1$ on $f(W)$, and

(iii) for each $\mathbf{y} \in f(W)$,

$$D(f^{-1})(\mathbf{y}) = [Df(f^{-1}(\mathbf{y}))]^{-1},$$

where $[\,]^{-1}$ represents matrix inversion (see Theorem C.5).

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Remark (11.42)

The hypothesis "$\Delta_f \neq 0$ in Theorem 11.39 can be relaxed.

Proof.

If $f(x) = x^3$, then $f : \mathbf{R} \to \mathbf{R}$ and its inverse $f^{-1}(x) = \sqrt[3]{x}$ are continuous on $\mathbf{R}$, but $\Delta_f(0) = f'(0) = 0$

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Remark (11.43)

The hypothesis "$\Delta_f \neq 0$ in Theorem 11.41 cannot be relaxed. In fact, if $f : B_r(\mathbf{a}) \to \mathbf{R}^n$ is differentiable at $\mathbf{a}$ and its inverse f^{-1} exists and is differentiable at $f(\mathbf{a})$, then $\Delta_f(\mathbf{a}) \neq 0$.

Proof.

Suppose to the contrary that f is differentiable at $\mathbf{a}$ but $\Delta_f(\mathbf{a}) = 0$. By Exercise 8, p.338, and the Chain Rule,

$$I = D(f^{-1} \circ f)(\mathbf{a}) = D(f^{-1})(f(\mathbf{a}))Df(\mathbf{a}).$$

Taking the determinant of this identity, we have

$$1 = \Delta_{f^{-1}}(f(\mathbf{a}))\Delta_f(\mathbf{a}) = 0,$$

a contradiction.

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Remark (11.44)

The hypothesis "f is $\mathcal{C}^1$ on V" in Theorem 11.41 cannot be relaxed.

Proof.

If $f(x) = x + 2x^2 \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, then $f : \mathbf{R} \to \mathbf{R}$ is differentiable on $V := (-1, 1)$ and $f'(0) = 1 \neq 0$. However, since

$$f\left(\frac{2}{(4k-1)\pi}\right) < f\left(\frac{2}{(4k+1)\pi}\right) < f\left(\frac{2}{(4k-3)\pi}\right)$$

for $k \in \mathbf{N}$, f is not 1-1 on any open set that contains 0. Therefore, no open subset of $f(V)$ can be closen on which f^{-1} exists. □

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Theorem (11.47 The Implicit Function Theorem)

Suppose that V is open in $\mathbf{R}^{n+p}$, and $F = (F_1, \ldots, F_n) : V \to \mathbf{R}^n$ is $\mathcal{C}^1$ on V. Suppose further that $F(\mathbf{x}_0, \mathbf{t}_0) = \mathbf{0}$ for some $(\mathbf{x}_0, \mathbf{t}_0) \in V$, where $\mathbf{x}_0 \in \mathbf{R}^n$ and $\mathbf{t}_0 \in \mathbf{R}^p$. If

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}(\mathbf{x}_0, \mathbf{t}_0) \neq 0,$$

then there is an open set $W \subset \mathbf{R}^p$ containing $\mathbf{t}_0$ and a unique continuously differentiable function $g : W \to \mathbf{R}^n$ such that $g(\mathbf{t}_0) = \mathbf{x}_0$, and $F(g(\mathbf{t}), \mathbf{t}) = \mathbf{0}$ for all $\mathbf{t} \in W$.

Example (11.49)

Prove that there exist function $u, v : \mathbf{R}^4 \to \mathbf{R}$, continuously differentiable on some ball B centered at the point $(x, y, z, w) = (2, 1, -1, -2)$, such that $u(2, 1, -1, -2) = 4,\ v(2, 1, -1, -2) = 3$, and the equations

$$u^2 + v^2 + w^2 = 29, \qquad \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17$$

both hold for all (x, y, z, w) in B.

Proof.

Set $n = 2,\ p = 4$, and

$$F(u, v, x, y, z, w) = \left(u^2 + v^2 + w^2 - 29,\ \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} - 17\right).$$

Then $F(4, 3, 2, 1, -1, -2) = (0, 0)$, and

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \det \begin{bmatrix} 2u & 2v \\ \frac{2u}{x^2} & \frac{2v}{y^2} \end{bmatrix} = 4uv\left(\frac{1}{y^2} - \frac{1}{x^2}\right).$$

This determinant is nonzero when $u = 4$, $v = 3$, $x = 2$, and $y = 1$. Therefore, such functions u, v exist by the Implicit Function Theorem.

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