Advanced Calculus (II)

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11.7: Optimization

Definition (11.50)

Let V be open in $\mathbf{R}^n$, let $\mathbf{a} \in V$, and suppose that $f : V \to \mathbf{R}$.

(i) $f(\mathbf{a})$ is called a *local minimum* of f if and only if there is an $r > 0$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{a})$.

(ii) $f(\mathbf{a})$ is called a *local maximum* of f if and only if there is an $r > 0$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{a})$.

(iii) $f(\mathbf{a})$ is called a *local extremum* of f if and only if $f(\mathbf{a})$ is a local maximum or a local minimum of f.

Remark (11.51)

If the first-order partial derivatives of f exist at $\mathbf{a}$, and $f(\mathbf{a})$ is a local extremum of f, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

Remark (11.52)

There exist continuously differentiable functions that satisfy $\nabla f(\mathbf{a}) = \mathbf{0}$ such that $f(\mathbf{a})$ is neither a local maximum nor a local minimum.

Definition (11.53)

Let V be open in $\mathbf{R}^n$, let $\mathbf{a} \in V$, and let $f : V \to \mathbf{R}$ be differentiable at $\mathbf{a}$. Then $\mathbf{a}$ is call a *saddle point* of f if $\nabla f(\mathbf{a}) = \mathbf{0}$ and there is a $r_0 > 0$ such that given any $0 < \rho < r_0$ there are points $\mathbf{x}, \mathbf{y} \in B_\rho(\mathbf{a})$ that satisfy

$$f(\mathbf{x}) < f(\mathbf{a}) < f(\mathbf{y}).$$

Example (11.54)

Find the maximum and minimum of
$f(x, y) = x^2 - x + y^2 - 2y$ on $H = B_1(0, 0)$.

Lemma (11.55)

Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and $f : V \to \mathbf{R}$. If all second-order partial derivatives of f exist at $\mathbf{a}$ and $D^{(2)}f(\mathbf{a};\mathbf{h}) > 0$ for all $\mathbf{h} \neq \mathbf{0}$, then there is an $m > 0$ such that

$$D^{(2)}f(\mathbf{a};\mathbf{x}) \geq m\|\mathbf{x}\|^2 \tag{33}$$

for all $\mathbf{x} \in \mathbf{R}^n$.

Proof.

Set $H = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| = 1\}$ and consider the function

$$g(\mathbf{x}) := D^{(2)}f(\mathbf{a};\mathbf{x}) := \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 f}{\partial x_k \partial x_j}(\mathbf{a})x_j x_k, \quad \mathbf{x} \in \mathbf{R}^n.$$

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Proof.

By hypothesis, g is continuous and positive on $\mathbf{R}^n \setminus \{\mathbf{0}\}$, hence on H. Since H is compact, it follows from the Extreme Value Theorem that g has a positive minimum m on H.

Clearly, (33) holds for $\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, then $\frac{\mathbf{x}}{\|\mathbf{x}\|} \in H$, and it follows from the choice of g and m that

$$D^{(2)} f(\mathbf{a}; \mathbf{x}) = \frac{g(\mathbf{x})}{\|\mathbf{x}\|^2} \|\mathbf{x}\|^2 = g\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \|\mathbf{x}\|^2 \geq m \|\mathbf{x}\|^2.$$

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Theorem (11.56 Second Derivative Test)

Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and suppose that $f : V \to \mathbf{R}$ satisfies $\nabla f(\mathbf{a}) = \mathbf{0}$. Suppose further that the second-order total differential of f exists on V and is continuous at $\mathbf{a}$.

(i) If $D^{(2)} f(\mathbf{a}; \mathbf{h}) > 0$ for all $\mathbf{h} \neq \mathbf{0}$, then $f(\mathbf{a})$ is a local minimum of f.

(ii) If $D^{(2)} f(\mathbf{a}; \mathbf{h}) < 0$ for all $\mathbf{h} \neq \mathbf{0}$, then $f(\mathbf{a})$ is a local maximum of f.

(iii) If $D^{(2)} f(\mathbf{a}; \mathbf{h})$ takes on both positive and negative values for $\mathbf{h} \in \mathbf{R}^n$, then $\mathbf{a}$ is a saddle point of f.

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(iii) If $D^{(2)}f(\mathbf{a}; \mathbf{h})$ takes on both positive and negative values for $\mathbf{h} \in \mathbf{R}^n$, then $\mathbf{a}$ is a saddle point of f.

Remark (11.57)

If $D^{(2)}f(\mathbf{a};\mathbf{h}) \geq 0$, then $f(\mathbf{a})$ can be a local minimum or $\mathbf{a}$ can be a saddle point.

Lemma (11.58)

Let $A, B, C \in \mathbf{R}$*,* $D = B^2 - AC$*, and* $\phi(h,k) = Ah^2 + 2Bhk + Ck^2$*.*

(i) If $D < 0$*, then* A *and* $\phi(h,k)$ *have the same sign for all* $(h,k) \neq (0,0)$*.*

(ii) If $D > 0$*, then* $\phi(h,k)$ *takes on both positive and negative values as* (h,k) *varies over* $\mathbf{R}^2$*.*

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Theorem (11.59)

Let V be open in $\mathbf{R}^2$, $(a,b) \in V$, and suppose that $f : V \to \mathbf{R}$ satisfies $\nabla f(a,b) = \mathbf{0}$. Suppose further that the second-order total differential of f exists on V and is continuous at (a,b), and set

$$D = f_{xy}^2(a,b) - f_{xx}(a,b) f_{yy}(a,b).$$

(i) If $D < 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(ii) If $D < 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(iii) If $D > 0$, then (a,b) is a saddle point.

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Remark (11.60)

If the discriminant $D = 0$, $f(a, b)$ may be a local maximum, a local minimum, or (a, b) may be a saddle point.

Definition (11.61)

Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and $f, g_j : V \to \mathbf{R}$ for $j = 1, 2, \ldots, m$.

(i) $f(\mathbf{a})$ is called a *local minimum of f subject to constraints* $g_j(\mathbf{a}) = 0$, $j = 1, \ldots, m$, if and only if there is a $\rho > 0$ such that $\mathbf{x} \in B_\rho(\mathbf{a})$ and $g_j(\mathbf{x}) = 0$ for all $j = 1, \ldots, m$ imply $f(\mathbf{x}) \geq f(\mathbf{a})$.

(ii) $f(\mathbf{a})$ is called a *local maximum of f subject to constraints* $g_j(\mathbf{a}) = 0$, $j = 1, \ldots, m$, if and only if there is a $\rho > 0$ such that $\mathbf{x} \in B_\rho(\mathbf{a})$ and $g_j(\mathbf{x}) = 0$ for all $j = 1, \ldots, m$ imply $f(\mathbf{x}) \leq f(\mathbf{a})$.

Theorem (11.63 Lagrange Multipliers)

Let $m < n$, V be open in $\mathbf{R}^n$ and $f, g_j : V \to \mathbf{R}$ be $\mathcal{C}^1$ on V for $j = 1, 2, \ldots, m$. Suppose that there is an $\mathbf{a} \in V$ such that

$$\frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)}(\mathbf{a}) \neq 0.$$

If $f(\mathbf{a})$ is a local extremum of f subject to constraints $g_k(\mathbf{a}) = 0$, $k = 1, \ldots, m$, then there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\nabla f(\mathbf{a}) + \sum_{k=1}^{m} \lambda_k \nabla g_k(\mathbf{a}) = \mathbf{0}. \tag{36}$$

Example (11.64)

Find all extrema of $x^2 + y^2 + z^2$ subject to the constraints $x - y = 1$ and $y^2 - z^2 = 1$.

Thank you.