Advanced Calculus (II)

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12.3: Iterate Integral

We call

$$\int_c^d \int_a^b f(x_1, \ldots, x_n) dx_j dx_k := \int_c^d \left(\int_a^b f(x_1, \ldots, x_n) dx_j \right) dx_k$$

an iterated integral when the integrals on the right side exist.

Lemma (12.30)

Let $R = [a, b] \times [c, d]$ be a two-dimensional rectangle and suppose that $f : R \to \mathbf{R}$ is bounded. If $f(x, \cdot)$ is integrable on $[c, d]$, then

$$(22) \quad (L) \iint_R f\, dA \le (L) \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

$$\le (U) \int_a^b \left(\int_c^d f(x, y) dy \right) dx \le (U) \iint_R f\, dA.$$

Proof.

Let $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, where $\{x_0, \dots, x_k\}$ is a partition of $[a, b]$ and $\{y_0, \dots, y_\ell\}$ is a partition of $[c, d]$. Then $\mathcal{G} = \{R_{ij} : i = 1, 2, \dots, k,\ j = 1, 2, \dots, \ell\}$ is a grid on R.

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Proof.

Let $\varepsilon > 0$, choose $\mathcal{G}$ so that

$$U(f,\mathcal{G})-\varepsilon < (U)\iint_R f\,dA, \tag{23}$$

and set

$$M_{ij} = \sup_{(x,y)\in R_{ij}} f(x,y). \tag{24}$$

Since $(U)\int_a^b \phi(x)dx = \sum_{i=1}^{k}(U)\int_{x_{i-1}}^{x_i} \phi(x)dx$ and

$$(U)\int_a^b (\phi(x)+\psi(x))dx \le (U)\int_a^b \phi(x)dx + (U)\int_a^b \psi(x)dx$$

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for any bounded function ϕ and ψ defined on $[a, b]$ (see Exercise 7, p.116), we can write

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\begin{aligned}
&(U)\int_a^b \left(\int_c^d f(x,y)dy \right) dx \\
&= \sum_{i=1}^{k} (U)\int_{x_{i-1}}^{x_i} \left(\sum_{j=1}^{\ell} \int_{y_{j-1}}^{y_j} f(x,y)dy \right) dx \\
&\le \sum_{i=1}^{k} \sum_{j=1}^{\ell} (U)\int_{x_{i-1}}^{x_i} \left(\int_{y_{j-1}}^{y_j} f(x,y)dy \right) dx \\
&\le \sum_{i=1}^{k} \sum_{j=1}^{\ell} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = U(f, \mathcal{G}).
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Proof.

It follows from (23) that

$$(U)\int_a^b\left(\int_c^d f(x,y)dy\right)dx < (U)\iint_R f\,dA+\varepsilon.$$

Taking the limit of this inequality as $\varepsilon\to 0$, we obtain

$$(U)\int_a^b\left(\int_c^d f(x,y)dy\right)dx \le (U)\iint_R f\,dA.$$

Similarly,

$$(L)\int_a^b\left(\int_c^d f(x,y)dy\right)dx \ge (L)\iint_R f\,dA.$$

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Theorem (12.31 Fubini's Theorem)

Let $R = [a, b] \times [c, d]$ be a two-dimensional rectangle and let $f : R \to \mathbf{R}$. Suppose that $f(x, \cdot)$ is integrable on $[c, d]$ for each $x \in [a, b]$, that $f(\cdot, y)$ is integrable on $[a, b]$ for each $y \in [c, d]$, and that f is integrable on R (as a function of two variables). Then

$$\iint_R f\, dA = \int_a^b \int_c^d f(x, y) dydx = \int_c^d \int_a^b f(x, y) dxdy.$$

Note: These hypotheses hold if f is continuous on the rectangle $[a, b] \times [c, d]$.

Proof.

For each $x \in [a, b]$, set $g(x) = \int_c^d f(x, y)dy$. Since f is integrable on R, Lemma 12.30 implies

$$\iint_R f\, dA = (U) \int_a^b g(x)dx = (L) \int_a^b g(x)dx.$$

Hence, g is integrable on $[a, b]$ and the first identity in (25) holds. Reversing the roles of x and y, we obtain

$$\iint_R f\, dA = \int_c^d \int_a^b f(x, y)dx\, dy.$$

Hence, the second identity in (25) holds. □

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Example (12.32)

Find

$$\int_0^1 \int_0^1 y^3 e^{xy^2} dydx.$$

Remark (12.33)

There exists a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $f(x, \cdot)$ and $f(\cdot, y)$ are both integrable on $[0, 1]$, but iterated integrals are not equal.

Proof.

Set

$$f(x,y) = \begin{cases} 2^{2n} & (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}), \ n \in \mathbf{N}, \\ -2^{2n+1} & (x,y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}), \ n \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

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Notice that for each fixed $y_0 \in [0, 1)$, $f(x, y_0)$ takes on only two nonzero values and is integrable on $[0, 1)$ in x. For example, if $y_0 \in [2^{-n}, 2^{-n+1})$, then $f(x, y_0) = 2^{2n}$ for $x \in [2^{-n}, 2^{-n+1})$, and $f(x, y_0) = -2^{2n+1}$ for $x \in [2^{-n-1}, 2^{-n})$; hence, $f(x, y_0)$ is bounded on $[0, 1)$, and

$$\int_0^1 f(x, y_0)dx = \int_{2^{-n}}^{2^{-n+1}} 2^{2n}dx - \int_{2^{-n-1}}^{2^{-n}} 2^{2n+1}dx \tag{26}$$

$$= 2^n - 2^n = 0.$$

the same is true for $f(x_0, y)$ when $x_0 \in [0, \frac{1}{2})$, but when $x_0 \in [\frac{1}{2}, 1)$, $f(x_0, y)$ takes on only nonzero value, namely, $f(x_0, y) = 4$ when $y \in [\frac{1}{2}, 1)$, and equals zero otherwise. It follows that

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Notice that for each fixed $y_0 \in [0,1)$, $f(x,y_0)$ takes on only two nonzero values and is integrable on $[0,1)$ in x. For example, if $y_0 \in [2^{-n}, 2^{-n+1})$, then $f(x,y_0) = 2^{2n}$ for $x \in [2^{-n}, 2^{-n+1})$, and $f(x,y_0) = -2^{2n+1}$ for $x \in [2^{-n-1}, 2^{-n})$; hence, $f(x,y_0)$ is bounded on $[0,1)$, and

$$(26) \quad \int_0^1 f(x,y_0)dx = \int_{2^{-n}}^{2^{-n+1}} 2^{2n}dx - \int_{2^{-n-1}}^{2^{-n}} 2^{2n+1}dx$$

$$= 2^n - 2^n = 0.$$

the same is true for $f(x_0,y)$ when $x_0 \in [0,\frac{1}{2})$, but when $x_0 \in [\frac{1}{2},1)$, $f(x_0,y)$ takes on only nonzero value, namely, $f(x_0,y) = 4$ when $y \in [\frac{1}{2},1)$, and equals zero otherwise. It follows that

□

Proof.

$$\int_0^1 \int_0^1 f(x,y)dy\,dx = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 4dy\,dx = 1.$$

On the other hand, by (26) we have

$$\int_0^1 \int_0^1 f(x,y)dx\,dy = 0.$$

Thus the iterated integrals of f are not equal. (Of course, by Fubini's Theorem, f itself cannot be Riemann integrable on $[0,1) \times [0,1)$. In fact, f is not even bounded.) □

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Lemma (12.36)

Let $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ *be an n-dimensional rectangle and let* $f : R \to \mathbf{R}$ *be integrable on R. If, for each* $\mathbf{x} := (x_1, \ldots, x_{n-1}) \in R_n := [a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]$*, the function* $f(\mathbf{x}.\cdot)$ *is integrable on* $[a_n, b_n]$*, then*

$$\int_{a_n}^{b_n} f(\mathbf{x}, t)dt$$

is integrable on R_n*, and*

$$\int_R f(\mathbf{x}, t)d(\mathbf{x}, t) = \int_{R_n} \int_{a_n}^{b_n} f(\mathbf{x}, t)dt\, d\mathbf{x}. \tag{27}$$

Proof.

By repeating the argument of Lemma 12.30, we have

$$
\begin{aligned}
(L)\int_R f(\mathbf{x},t)d(\mathbf{x},t) &\le (L)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_R f(\mathbf{x},t)d(\mathbf{x},t)
\end{aligned}
$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

Proof.

By repeating the argument of Lemma 12.30, we have

$$
\begin{aligned}
(L)\int_R f(\mathbf{x},t)d(\mathbf{x},t) &\le (L)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_R f(\mathbf{x},t)d(\mathbf{x},t)
\end{aligned}
$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

Proof.

By repeating the argument of Lemma 12.30, we have

$$(L)\int_R f(\mathbf{x},t)d(\mathbf{x},t) \le (L)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x}$$

$$\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x}$$

$$\le (U)\int_R f(\mathbf{x},t)d(\mathbf{x},t)$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

Proof.

By repeating the argument of Lemma 12.30, we have

$$
\begin{aligned}
(L)\int_R f(\mathbf{x},t)d(\mathbf{x},t) &\le (L)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_R f(\mathbf{x},t)d(\mathbf{x},t)
\end{aligned}
$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

Proof.

By repeating the argument of Lemma 12.30, we have

$$
\begin{aligned}
(L)\int_R f(\mathbf{x},t)\,d(\mathbf{x},t) &\le (L)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)\,dt\,d\mathbf{x} \\
&\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)\,dt\,d\mathbf{x} \\
&\le (U)\int_R f(\mathbf{x},t)\,d(\mathbf{x},t)
\end{aligned}
$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

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&\le (U)\int_{R_n}\int_{a_n}^{b_n} f(\mathbf{x},t)dt\,d\mathbf{x} \\
&\le (U)\int_R f(\mathbf{x},t)d(\mathbf{x},t)
\end{aligned}
$$

for any bounded f. Since f is integrable on R, it follows that (27) holds.

□

Theorem (12.39)

Let E be a projectable region in $\mathbf{R}^n$ generated by j, H, ϕ, and ψ. Then E is a Jordan region in $\mathbf{R}^n$. Moreover, if $f : E \to \mathbf{R}$ is continuous on E, then

$$(28) \quad \int_E f(\mathbf{x})d\mathbf{x}$$

$$= \int_H \left(\int_{\phi(x_1,\ldots,\widehat{x}_j,\ldots,x_n)}^{\psi(x_1,\ldots,\widehat{x}_j,\ldots,x_n)} f(x_1,\ldots,x_n)dx_j \right) d(x_1,\ldots,\widehat{x}_j,\ldots,x_n).$$

Thank you.