Advanced Calculus (I)

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4.1 The Derivative

Definition

A real function f is said to be *differentiable* at a point $a \in \mathbf{R}$ if and only if f is defined on some open interval I containing a and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$

exists. In this case $f'(a)$ is called the *derivative* of f at a.

Notation: $D_x f = \frac{df}{dx} = f^{(1)} = f'$

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Theorem

A real function f is differentiable at some point $a \in \mathbf{R}$ if and only if there exists an open interval I and a function $F : I \to \mathbf{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and

$$f(x) = F(x)(x - a) + f(a) \tag{3}$$

holds for all $x \in I$, in which case $F(a) = f'(a)$.

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Let $f : \mathbf{R} \to \mathbf{R}$. Then f is differentiable at a if and only if there is a function T of the form $T(x) := mx$ such that

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0. \tag{4}$$

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Let I be a nondegenerate interval.

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A function $f : I \to \mathbf{R}$ is said to be *differentiable* on I if and only if

$$f'_I(a) = \lim_{\substack{x \to a \\ x \in I}} \frac{f(x) - f(a)}{x - a}$$

exists and is finite for every $a \in I$.

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f is said to be *continuously differentiable* on I if and only if f'_I exists and is continuous on I.

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Example:

The function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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Remark:

$f(x) = |x|$ is differentiable on $[0, 1]$ and on $[-1, 0]$ but not on $[-1, 1]$.

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Proof:

Since $f(x) = x$ when $x > 0$ and $= -x$ when $x < 0$, it is clear that f is differentiable on $[-1,0) \cup (0,1]$ (with $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$). By example 4.5, f is not differentiable at $x = 0$. However,

$$f'_{[0,1]}(0) = \lim_{h \to 0+} \frac{|h|}{h} = 1 \quad and \quad f'_{[-1,0]}(0) = \lim_{h \to 0-} \frac{|h|}{h} = 1$$

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