# Advanced Calculus (I)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

**Definition**

Let $a, b \in \mathbf{R}$ with $a < b$.

(i)

A *partition* of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

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The *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

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Let $a, b \in \mathbf{R}$ with $a < b$, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval [a,b], and suppose that $f : [a, b] \to \mathbf{R}$ is bounded.

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The *upper Riemann sum* of f over P is the number

$$U(f, P) := \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}),$$

where

$$M_j(f) := \sup_{x \in [x_{j-1}, x_j]} f(x)$$

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Let $a, b \in \mathbf{R}$ with $a < b$. A function $f : [a, b] \to \mathbf{R}$ is said to be *(Riemann) integrable* on [a,b], if and only if f is bounded on [a,b] and for every $\epsilon > 0$ there is a partition P of [a,b] such that $U(f, P) - L(f, P) < \epsilon$.

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## Theorem

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## Proof:

Let $\epsilon > 0$. Since f is uniformly continuous on [a,b], choose $\delta > 0$ such that

$$(1) \qquad |x-y| < \delta \ \textit{ implies } \ |f(x)-f(y)| < \frac{\epsilon}{b-a}.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of [a,b] that satisfies $||P|| < \delta$. Fix an index j and notice, by the Extreme Value Theorem, that there are points $x_m$ and $x_M$ in $[x_{j-1}, x_j]$ such that

$$f(x_m) = m_j(f) \ \textit{ and } \ f(x_M) = M_j(f).$$

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$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) < \frac{\epsilon}{b-a} \sum_{j=1}^{n} (x_j - x_{j-1}$$

(The last step comes from telescoping.) □

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**Example**:

The function

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

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Let $a, b \in \mathbf{R}$ with $a < b$, and $f : [a, b] \to \mathbf{R}$ be bounded.

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The *upper integral* of f on [a,b] is the number

$$(U) \int_a^b f(x)dx := \inf\{U(f, P) : P \ \ is \ \ a \ \ partition \ \ of \ \ [a, b]\}$$

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If the upper and lower integrals of f on [a,b] are equal, we define the integral of f on [a,b] to be the common value

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**Remark:**

If $f : [a, b] \to \mathbf{R}$ is bounded, then its upper and lower integrals exists and are infinite, and satisfy

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