Advanced Calculus (I)

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6.1 Introduction

Definition

Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series whose terms a_k belong to $\mathbf{R}$

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The *partial sums of S of order n* are the numbers defined, for each $n \in \mathbf{N}$, by

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S is said to *converge* if and only if its sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbf{R}$ as $n \to \infty$; i.e., for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|s_n - s| < \epsilon$. In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

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S is said to *diverge* if and only if its sequence of partial sums $\{s_n\}$ does not converge as $n \to \infty$. When s_n diverges to $+\infty$ as $n \to \infty$, we shall also write

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Example: [Harmonic Series]

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The sequence $1/k$ converges to zero (by Example 2.2). On the other hand, by the Comparison Theorem for Integrals,

$$\sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x} dx = \int_{1}^{n+1} \frac{1}{x} dx = \log(n+1).$$

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Theorem (Divergence Test)

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. If a_k does not converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

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Suppose to the contrary that $\sum_{k=1}^{\infty} a_k$ converges to some $s \in \mathbf{R}$. By definition, the sequence of partial sums $s_n := \sum_{k=1}^{n} a_k$ converges to s as $n \to \infty$. Therefore, $a_k = s_k - s_{k-1} \to s - s = 0$ as $k \to \infty$, a contradiction. $\square$

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Theorem (Geometric Series)

The series $\sum_{k=1}^{\infty} x^k$ *converges if and only if* $|x| < 1$, *in which case*

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(see also Exercise 1.)

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Proof:

If $|x| \geq 1$, then $\sum_{k=1}^{\infty} x^k$ diverges by the Divergence Test. If $|x| < 1$, then set $s_n = \sum_{k=1}^{n} x^k$ and observe by the telescoping that

$$\begin{aligned} (1-x)S_n &= (1-x)(x + x^2 + \cdots + x^n) \\ &= x + x^2 + \cdots + x^n - x^2 - x^3 - \cdots - x^{n+1} \\ &= x - x^{n+1}. \end{aligned}$$

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Hence,

$$s_n = \frac{x}{1-x} - \frac{x^{n+1}}{1-x}$$

for all $n \in \mathbf{N}$. Since $x^{n+1} \to 0$ as $n \to \infty$ for all $|x| < 1$ (see Example 2.20), we conclude that $s_n \to \frac{x}{(1-x)}$ as $n \to \infty$. $\square$

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Theorem (Cauchy Criterion)

Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

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Let s_n represent the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ and set $s_0 = 0$. By Cauchy's Theorem (Theorem 2.29), s_n converges if and only if given $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that $m, n \geq N$ imply $|s_m - s_{n-1}| < \epsilon$. Since

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Let $\{a_k\}$ and $\{b_k\}$ be real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

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Thank you.