Advanced Calculus (I)

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7.1 Uniform convergence of sequences

Definition

Let E be a nonempty subset of $\mathbf{R}$. A sequence of functions $f_n : E \to \mathbf{R}$ is said to *converge pointwise* on E (notation: $f_n \to f$ pointwise on E as $n \to \infty$) if and only if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$.

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Let $f_n(x) = x^n$ and set

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise on [0,1] (see Example 2.20), each f_n is continuous and differentiable on [0,1]. but f is neither differentiable nor continuous at $x = 1$. $\square$

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$$f_n(x) = \begin{cases} 1 & x = \frac{p}{m} \in Q, \text{ written in reduced form, where } m \leq n \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbf{N}$ and

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There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on [0,1] but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)' \quad (1)$$

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Proof:

Let $f_1(x) = 1$, and for $n > 1$ let f_n be a sequence of functions whose graphs are triangles with bases $\frac{2}{n}$ and altitudes n (see Figure 7.1). By the pointslope form, formulas for these $f_n's$ can be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1. \end{cases}$$

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Then $f_n \to 0$ pointwise on [0,1], and since the area of a trianlglge is one-half base times altitude, $\int_0^1 f_n(x)dx = 1$ for each $n \in \mathbf{N}$. Thus, the left side of (2) is 1 but the right side is zero. □

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Let E be a nonempty subset of **R**. A sequence of functions $f_n : E \to \mathbf{R}$ is said to *converge uniformly* on E to a function f (notation: $f_n \to f$ uniformly on E as $n \to \infty$) if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

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Let E be a nonempty subset of $\mathbf{R}$ and suppose that $f_n \to f$ uniformly on E. If each f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \in E$.

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Suppose that $|x - x_0| < \delta$ and $x \in E$. Then

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Suppose that $f_n \to f$ uniformly on a closed interval [a,b]. If each f_n is integrable on [a,b], then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx.$$

In fact, $\lim_{n\to\infty} \int_a^x f_n(t)dt = \int_a^x f(t)dt$ *uniformly for* $x \in [a,b]$.

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By Exercise 5, f is bounded on [a,b]. To prove that f is integrable, let $\epsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$(3) \qquad n \geq N \ \ implies \ \ |f(x) - f_n(x)| < \frac{\epsilon}{3(b-a)}$$

for all $x \in [a, b]$. Using this inequality for $n = N$, we see that by the definition of upper and lower sums,

$$U(f - f_N, P) \leq \frac{\epsilon}{3} \ \ and \ \ L(f - f_N, P) \geq -\frac{\epsilon}{3}$$

for any partition P of [a,b].

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It follows that

$$\begin{aligned} & U(f, P) - L(f, P) \\ \leq \ & U(f - f_N, P) + U(f_N, P) - L(f_N, P) - L(f - f_N, P) \\ < \ & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon; \end{aligned}$$

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$$\begin{aligned} & U(f, P) - L(f, P) \\ \leq\ & U(f - f_N, P) + U(f_N, P) - L(f_N, P) - L(f - f_N, P) \\ <\ & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon; \end{aligned}$$

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Lemma: [Uniform Cauchy Criterion]

Let E be a nonempty subset of $\mathbf{R}$ and let $f_n : E \to \mathbf{R}$ be a sequence of functions. Then f_n converges uniformly on E if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

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Proof:

Suppose first that $f_n \to f$ uniformly on E as $n \to \infty$. Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \ \ implies \ \ |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for $x \in E$. Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$, it is clear that (4) holds for all $x \in E$.

Conversely, if (4) holds for $x \in E$, then $\{f_n(x)\}_{n \in \mathbf{N}}$ is Cauchy for each $x \in E$.

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Hence, by Cauchy Theorem for sequence (Theorem 2.29),

$$f(x) := \lim_{n\to\infty} f_n(x)$$

exists for each $x \in E$. Take the limit of the second inequality in (4) as $m \to \infty$. We obtain $|f_n(x) - f(x)| \le \epsilon$ for all $n \ge N$ and $x \in E$. Hence, by definition, $f_n \to f$ uniformly on E. □

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Theorem

Let (a,b) be a bounded interval and suppose that f_n is a sequence of functions that converges at some $x_0 \in (a,b)$. If each f_n is differentiable on (a,b), and $f_n{}'$ converges uniformly on (a,b) as $n \to \infty$, then f_n converges uniformly on (a,b) and

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'$$

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Thank you.