Advanced Calculus (II)

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8.1: Algebraic Structure of $\mathbf{R}^n$

Definition (8.1)

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$ be vectors and $\alpha \in \mathbf{R}$ be a scalar.

(i) The *sum* of $\mathbf{x}$ and $\mathbf{y}$ is the vector

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

(ii) The *difference* of $\mathbf{x}$ and $\mathbf{y}$ is the vector

$$\mathbf{x} - \mathbf{y} := (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

Definition (8.1)

(iii) The product of a scalar α and a vector $\mathbf{x}$ is the vector

$$\alpha\mathbf{x} := (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

(iv) The (*Euclidean*) *dot product* (or *scalar product* or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$ is the scalar

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Theorem (8.2)

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$ *and* $\alpha, \beta \in \mathbf{R}$. *Then*

$\alpha\mathbf{0} = \mathbf{0}$,

$0\mathbf{x} = \mathbf{0}$,

$1\mathbf{x} = \mathbf{x}$,

$\alpha(\beta\mathbf{x}) = \beta(\alpha\mathbf{x}) = (\alpha\beta)\mathbf{x}$,

$\alpha(\mathbf{x} \cdot \mathbf{y}) = (\alpha\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\alpha\mathbf{y})$,

$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$,

$\mathbf{0} + \mathbf{x} = \mathbf{x}$,

$\mathbf{x} - \mathbf{x} = \mathbf{0}$,

$\mathbf{0} \cdot \mathbf{x} = 0$,

$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$,

$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$,

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$,

$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

Definition (8.3)

Let **a** and **b** be nonzero vectors in $\mathbf{R}^n$.
(i) **a** and **b** are said to be *parallel* if and only if these is a scalar $t \in R$ such that $\mathbf{a} = t\mathbf{b}$.

(ii) **a** and **b** are said to be *orthogonal* if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Definition (8.4)

Let $\mathbf{x} \in \mathbf{R}^n$.

(i) The (*Euclidean*) *norm* (or *magnitude*) of $\mathbf{x}$ is the scalar

$$\|\mathbf{x}\| := \sqrt{\sum_{k=1}^{n} |x_k|^2}.$$

(ii) The $\ell^1 - norm$ (read L-one-norm) of $\mathbf{x}$ is the scalar

$$\|\mathbf{x}\|_1 := \sum_{k=1}^{n} |x_k|.$$

(iii) The $sup - norm$ of $\mathbf{x}$ is the scalar

$$\|\mathbf{x}\|_\infty := max\{|x_1|, \ldots, |x_n|\}.$$

Theorem (8.5 Cauchy-Schwarz Inequality)

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, *then*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\| \, .$$

Proof.

The Cauchy-Schwarz Inequality for trivial when $\mathbf{y} = \mathbf{0}$. If $\mathbf{y} \neq \mathbf{0}$, substitute $t = (\mathbf{x} \cdot \mathbf{y}) / \|\mathbf{y}\|^2$ into

$$0 \leq \|\mathbf{x} - t\mathbf{y}\|^2 = (\mathbf{x} - t\mathbf{y}) \cdot (\mathbf{x} - t\mathbf{y}) = \|\mathbf{x}\|^2 - 2t(\mathbf{x} \cdot \mathbf{y}) + t^2 \|\mathbf{y}\|^2$$

to obtain

$$0 \leq \|\mathbf{x}\|^2 - t(\mathbf{x} \cdot \mathbf{y}) = \|\mathbf{x}\|^2 - \frac{(\mathbf{x} \cdot \mathbf{y})^2}{\|\mathbf{y}\|^2}.$$

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$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \text{ and } \|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|.$$

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This establishes the first inequality in (iii). By modifying the proof of Theorem 1.7, we can also establish the second inequality in (iii).

□

Remark (8.7)

Let $\mathbf{x} \in \mathbf{R}^n$.Then
(i) $|x_j| \leq \|\mathbf{x}\| \leq \sqrt{n}\,\|\mathbf{x}\|_\infty$ for each j = 1,2,. . . ,n,

(ii) $\|\mathbf{x}\| \leq \|\mathbf{x}\|_1$.

Proof.

(i) Let $1 \leq j \leq n$. By definition,

$$|x_j|^2 \leq \|\mathbf{x}\|^2 = x_1^2 + \ldots + x_n^2 \leq n(\max_{1\leq \ell \leq n} |x_\ell|)^2 = n\,\|\mathbf{x}\|_\infty^2$$

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(ii) Observe that

$$(|x_1| + \cdots + |x_n|)^2 = |x_1|^2 + \cdots + |x_n|^2 + 2 \sum_{(i,j \in A)} |x_i||x_j|$$

where $A = \{(i,j) : 1 \leq i,j \leq n \text{ and } i < j\}$. Since $\sum_{(i,j) \in A} |x_i||x_j| \geq 0$, we conclude that

$$\|\mathbf{x}\|^2 = |x_1|^2 + \ldots + |x_n|^2 \leq (|x_1| + \ldots + |x_n|)^2$$

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Definition (8.8)

The *cross product* of two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in R^3 is the vector defined by

$$\mathbf{x} \times \mathbf{y} := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

using the usual basis $i = e_1, j = e_2, k = e_3$, and the determinant operator (see Appendix C),we can give the cross product a more easily remembered form:

$$det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Theorem (8.9)

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^3$ *be vectors and* α *be a scalar. Then*

(*i*) $$\mathbf{x} \times \mathbf{x} = \mathbf{0}, \quad \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x},$$

(*ii*) $$(\alpha\mathbf{x}) \times \mathbf{y} = \alpha(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times (\alpha\mathbf{y}),$$

(*iii*) $$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z}),$$

(*iv*) $$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix},$$

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Proof.

These properties follow immediately from the definition. We will prove properties (iv),(v),and (vii) and leave the rest as an exercise.

(iv) Notice that by definition,

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = (x_2y_3-x_3y_2)z_1+(x_3y_1-x_1y_3)z_2+(x_1y_2-x_2y_1)z_3$$

$$= x_1(y_2z_3-y_3z_2)+x_2(y_3z_1-y_1z_3)+x_3(y_1z_2-y_2z_1).$$

Since this last expression is both the scalar $\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})$ and the value of the determinant on the right side of (iv) (expanded along the first row), this verifies (iv).

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Proof.

(v) Since $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$

$= (x_1, x_2, x_3) \times (y_2 z_3 - y_3 z_2, y_3 z_1 - y_1 z_3, y_1 z_2 - y_2 z_1)$,

the component of $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ is

$$x_2 y_1 z_2 - x_2 y_2 z_1 - x_3 y_3 z_1 + x_3 y_1 z_3$$

$$= (x_1 z_1 + x_2 z_2 + x_3 z_3) y_1 - (x_1 z_1 + x_2 z_2 + x_3 z_3) z_1.$$

this proves that the first conponents of $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ and $(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ are equal. A similar argument shows that the second and third components are also equal.

(vii) By parts (i) and (iv),

$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = -(\mathbf{y} \times \mathbf{x}) \cdot \mathbf{x} = -\mathbf{y} \cdot (\mathbf{x} \times \mathbf{x}) = -\mathbf{y} \cdot \mathbf{0} = 0$.

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$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = -(\mathbf{y} \times \mathbf{x}) \cdot \mathbf{x} = -\mathbf{y} \cdot (\mathbf{x} \times \mathbf{x}) = -\mathbf{y} \cdot \mathbf{0} = \mathbf{0}$.
Thus $(\mathbf{x} \times \mathbf{y})$ is orthogonal to $\mathbf{x}$. A similar calculation shows that $(\mathbf{x} \times \mathbf{y})$ is orthogonal to $\mathbf{y}$.

Proof.

(v) Since $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$
$= (x_1, x_2, x_3) \times (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1)$,
the component of $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ is

$$x_2y_1z_2 - x_2y_2z_1 - x_3y_3z_1 + x_3y_1z_3$$
$$= (x_1z_1 + x_2z_2 + x_3z_3)y_1 - (x_1z_1 + x_2z_2 + x_3z_3)z_1.$$

this proves that the first conponents of $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ and $(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ are equal. A similar argument shows that the second and third components are also equal.

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□

Remark (8.10)

Let **x** and **y** be nonzero vectors in $\mathbf{R}^3$ and θ be the angle between **x** and **y**. Then $\|\mathbf{x}\times\mathbf{y}\| = \|\mathbf{x}\|\,\|\mathbf{y}\|\sin\theta$.

Proof.

By theorem 8.9(vi) and $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$,

$$\|\mathbf{x}\times\mathbf{y}\|^2 = (\|\mathbf{x}\|\,\|\mathbf{y}\|)^2 - (\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta)^2$$

$$= (\|\mathbf{x}\|\,\|\mathbf{y}\|)^2(1-\cos^2\theta) = (\|\mathbf{x}\|\,\|\mathbf{y}\|)^2\sin^2\theta.$$

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Thank you.