# Advanced Calculus (II)

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2009

## 8.3: Topology Of $\mathbf{R}^n$

**Definition (8.19)**

Let $\mathbf{a} \in \mathbf{R}^n$.
(i) For each $r > 0$, the open ball centered at $\mathbf{a}$ of radius $r$ is the set of points

$$B_r(\mathbf{a}) := \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\}.$$

(ii) For each $r \geq 0$, the closed ball centered at $\mathbf{a}$ of radius $r$ is the set of points

$$\{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \mathbf{a}\| \leq r\}.$$

## Definition (8.20)

Let $n \in \mathbf{N}$.

(i) A set $V$ in $\mathbf{R}^n$ is said to be *open* if and only if for every $\mathbf{a} \in V$ there is an $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{a}) \subseteq V$.

(ii) A set $E$ in $\mathbf{R}^n$ is said to be *closed* if and only if $E^c := \mathbf{R}^n \backslash E$ is open.

**Remark (8.23)**

For each $n \in \mathbf{N}$, the empty set $\emptyset$ and the whole space $\mathbf{R}^n$ are both open and closed.

## Theorem (8.24)

*Let* $n \in \mathbf{N}$.

*(i) If* $\{V_\alpha\}_{\alpha \in A}$ *is any collection of open subsets of* $\mathbf{R}^n$, *then*

$$\bigcup_{\alpha \in A} V_\alpha$$

*is open.*

*(ii) If* $\{V_k : k = 1, 2, \ldots, p\}$ *is a finite collection of open subsets of* $\mathbf{R}^n$, *then*

$$\bigcap_{k=1}^{p} V_k := \bigcap_{k \in \{1,2,\ldots,p\}} V_k$$

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## Proof.

(i)Let $\mathbf{x} \in \bigcup_{\alpha \in A} V_\alpha$. Then $\mathbf{x} \in V_\alpha$ for some $\alpha \in A$. Since $V_\alpha$ is open, it follows that there is an $r > 0$ such that $B_r(\mathbf{x}) \subseteq V_\alpha$. Thus $B_r(\mathbf{x}) \subseteq \bigcup_{\alpha \in A} V_\alpha$; i.e., this union is open.

(ii) Let $\mathbf{x} \in \bigcap_{k=1}^{p} V_k$. Then $\mathbf{x} \in V_k$ for $k = 1, 2, \ldots, p$. Since each $V_k$ is open, it follows that there are numbers $r_k > 0$ such that $B_{r_k}(\mathbf{x}) \subseteq V_k$. Let $r = \min\{r_1, \ldots, r_p\}$. Then $r > 0$ and $B_r(\mathbf{x}) \subseteq V_k$ for all $k = 1, 2, \ldots, p$;i.e., $B_r(\mathbf{x}) \subseteq \bigcap_{k=1}^{p} V_k$. Hence this intersection is open.

(iii) By DeMorgan's Law (Theorem 1.41) and part (i),

$$\left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

is open, so $\bigcap_{\alpha \in A} E_\alpha$ is closed. □

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(iv) By DeMorgan's Law and part (ii),

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(v) Since $V \setminus E = V \cap E^c$ and $E \setminus V = E \cap V^c$, the former is open by part (ii), and the latter is closed by part (iii). □

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## Example

(i)

$$\bigcap_{k\in\mathbf{N}}(-\frac{1}{k},\frac{1}{k}) = \{0\} \textit{ is closed}.$$

(ii)

$$\bigcup_{k\in\mathbf{N}}[\frac{1}{k+1},\frac{k}{k+1}] = (0,1) \textit{ is open}.$$

## Definition (8.26)

Let $E \subseteq \mathbf{R}^n$.
(i) A set $U$ is said to be *relatively open* in $E$ if and only if there is an open set $A$ such that $U = E \cap A$.

(ii) A set $C$ is said to be *relatively closed* in $E$ if and only if there is a closed set $B$ such that $C = E \cap B$.

## Remark (8.27)

Let $U \subseteq E \subseteq \mathbf{R}^n$.
(i) Then $U$ is relatively open in $E$ if and only if for each $\mathbf{a} \in U$ there is an $r > 0$ such that $B_r(\mathbf{a}) \cap E \subset U$.

(ii) If $E$ is open, then $U$ is relatively open in $E$ if and only if $U$ is (plain old vanilla) open (in the usual sense).

## Definition (8.28)

Let $E$ be a subset of $\mathbf{R}^n$.
(i) A pair of sets $U$, $V$ is said to *separate* $E$ if and only if $U$ and $V$ are nonempty, relatively open in $E$,
$E = U \cup V$, *and* $U \cap V = \emptyset$.

(ii) $E$ is said to be *connected* if and only if $E$ cannot be separated by any pair of relatively open sets $U$, $V$.

**Remark (8.29)**

Let $E \subseteq \mathbf{R}^n$. If there exists a pair of open sets A, B such that $E \cap A \neq \emptyset$, $E \cap B \neq \emptyset$, $E \subseteq A \cup B$, *and* $A \cap B = \emptyset$, then $E$ is not connected.

**Theorem (8.30)**

*A subset $E$ of $\mathbf{R}$ is connected if and only if $E$ is an interval.*

*Thank you.*