# Advanced Calculus (II)

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## 8.4: Interior, Closure, and, Boundary

**Definition (8.31)**

Let $E$ be a subset of a Euclidean Space $\mathbf{R}^n$.
(i) The *interior* of $E$ is the set

$$E^o := \bigcup \{V : V \subseteq E \text{ and } V \text{ is open in } \mathbf{R}^n\}.$$

(ii) The *closure* of $E$ is the set

$$\overline{E} := \bigcap \{B : B \supseteq E \text{ and } B \text{ is closed in } \mathbf{R}^n\}.$$

## Theorem (8.32)

*Let $E \subseteq \mathbf{R}^n$. Then*

(i) $E^o \subseteq E \subseteq \overline{E}$,

(ii) *if $V$ is open and $V \subseteq E$ then $V \subseteq E^o$, and*

(iii) *if $C$ is closed and $C \supseteq E$ then $C \supseteq \overline{E}$.*

## Proof.

Since every open set $V$ in the union defining $E^o$ is a subset of $E$, it is clear that the union of these $V$'s is a subset of $E$. Thus $E^o \subseteq E$. A similar argument establishes $E \subseteq \overline{E}$. This proves (i).

By Definition 8.31, if $V$ is an open subset of $E$, then $V \subseteq E^o$ and if $C$ is a closed set containing $E$, then $\overline{E} \subseteq C$. This proves (ii) and (iii). □

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### Definition (8.34)

Let $E \subseteq \mathbf{R}^n$. The *boundary* of $E$ is the set $\partial E := \{\mathbf{x} \in \mathbf{R}^n :$ for all $r > 0$, $B_r(\mathbf{x}) \cap E \neq \emptyset$ and $B_r(\mathbf{x}) \cap E^c \neq \emptyset\}$.

## Theorem (8.36)

*Let* $E \subseteq \mathbf{R}^n$. *Then* $\partial E = \overline{E} \setminus E^o$.

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By Definition 8.34, it suffices to show that
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(10) $\mathbf{x} \in \overline{E}$ if and only if $B_r(\mathbf{x}) \cap E \neq \emptyset$ for all $r > 0$, and
(11) $\mathbf{x} \notin E^o$ if and only if $B_r(\mathbf{x}) \cap E^c \neq \emptyset$ for all $r > 0$.
We will provide the details for (10) and leave the proof of (11) as an exercise. Suppose that $\mathbf{x} \in \overline{E}$ but $B_{r_0}(\mathbf{x}) \cap E = \emptyset$ for some $r_0 > 0$. Then $(B_{r_0}(\mathbf{x}))^c$ is a closed set that contains $E$; hence, by theorem 8.32iii, $\overline{E} \subseteq (B_{r_0}(\mathbf{x}))^c$. It follows that $\overline{E} \cap B_{r_0}(\mathbf{x}) = \emptyset$, e.g., $\mathbf{x} \notin \overline{E}$, a contradiction. Conversely, suppose that $\mathbf{x} \notin \overline{E}$. Since $(\overline{E})^c$ is open, there is an $r_0 > 0$ such that $B_{r_0}(\mathbf{x}) \subseteq (\overline{E})^c$. In particular, $\emptyset = B_{r_0}(\mathbf{x}) \cap \overline{E} \supseteq B_{r_0}(\mathbf{x}) \cap E$ for some $r_0 > 0$.

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## Theorem (8.37)

*Let $A, B \subseteq \mathbf{R}^n$ Then*

(i) $(A \cup B)^o \supseteq A^o \cup B^o, \quad (A \cap B)^o = A^o \cap B^o,$

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(iii) $\partial(A \cup B) \subseteq \partial A \cup \partial B$, and $\partial(A \cap B) \subseteq \partial A \cup \partial B$.

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(i) Since the union of two open sets is open, $A^o \cup B^o$ is an open subset of $A \cup B$. Hence, by Theorem 8.32ii, $A^o \cup B^o \subseteq (A \cup B)^o$.

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(ii) Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it is clear that, by Theorem 8.32iii, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. To prove the reverse inequlity for union, suppose that $\mathbf{x} \notin \overline{A \cup B}$. Then, by, Definition 8.31, there is a closed set $E$ that contains $A \cup B$ such that $\mathbf{x} \notin E$. Since $E$ contains both $A$ and $B$, it follows that $\mathbf{x} \notin \overline{A}$ and $\mathbf{x} \notin \overline{B}$. This proves part(ii).

(iii) Let $\mathbf{x} \in \partial(A \cup B)$; i.e., suppose that $B_r(\mathbf{x})$ intersects $A \cup B$ and $(A \cup B)^c$ for all $r > 0$, Since $(A \cup B)^c = A^c \cap B^c$, it follows that $B_r(\mathbf{x})$ interscets both $A^c$ and $B^c$ for all $r > 0$. Thus $B_r(\mathbf{x})$ intersects $A$ and $A^c$ for all $r > 0$, or $B_r(\mathbf{x})$ intersects $B$ and $B^c$ for all $r > 0$; i.e., $\mathbf{x} \in \partial A \cup \partial B$. This proves the first set inequality in part (iii). A similar argument establishes the second inequality in part (iii).

## Proof.

(ii) Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it is clear that, by Theorem 8.32iii, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. To prove the reverse inequlity for union, suppose that $\mathbf{x} \notin \overline{A \cup B}$. Then, by, Definition 8.31, there is a closed set $E$ that contains $A \cup B$ such that $\mathbf{x} \notin E$. Since $E$ contains both $A$ and $B$, it follows that $\mathbf{x} \notin \overline{A}$ and $\mathbf{x} \notin \overline{B}$. This proves part(ii).

(iii) Let $\mathbf{x} \in \partial(A \cup B)$; i.e., suppose that $B_r(\mathbf{x})$ intersects $A \cup B$ and $(A \cup B)^c$ for all $r > 0$, Since $(A \cup B)^c = A^c \cap B^c$, it follows that $B_r(\mathbf{x})$ interscets both $A^c$ and $B^c$ for all $r > 0$. Thus $B_r(\mathbf{x})$ intersects $A$ and $A^c$ for all $r > 0$, or $B_r(\mathbf{x})$ intersects $B$ and $B^c$ for all $r > 0$; i.e., $\mathbf{x} \in \partial A \cup \partial B$. This proves the first set inequality in part (iii). A similar argument establishes the second inequality in part (iii).

## Proof.

(ii) Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it is clear that, by Theorem 8.32iii, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. To prove the reverse inequlity for union, suppose that $\mathbf{x} \notin \overline{A \cup B}$. Then, by, Definition 8.31, there is a closed set $E$ that contains $A \cup B$ such that $\mathbf{x} \notin E$. Since $E$ contains both $A$ and $B$, it follows that $\mathbf{x} \notin \overline{A}$ and $\mathbf{x} \notin \overline{B}$. This proves part(ii).

(iii) Let $\mathbf{x} \in \partial(A \cup B)$; i.e., suppose that $B_r(\mathbf{x})$ intersects $A \cup B$ and $(A \cup B)^c$ for all $r > 0$, Since $(A \cup B)^c = A^c \cap B^c$, it follows that $B_r(\mathbf{x})$ interscets both $A^c$ and $B^c$ for all $r > 0$. Thus $B_r(\mathbf{x})$ intersects $A$ and $A^c$ for all $r > 0$, or $B_r(\mathbf{x})$ intersects $B$ and $B^c$ for all $r > 0$; i.e., $\mathbf{x} \in \partial A \cup \partial B$. This proves the first set inequality in part (iii). A similar argument establishes the second inequality in part (iii). □

**Theorem (8.38)**

*Let $E \subseteq \mathbf{R}^n$. If there exist nonempty, relatively open sets $U, V$ which separate $E$, then there is a pair of open sets $A, B$ such that $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, $A \cap B = \emptyset$, and $E \subseteq A \cup B$.*

*Thank you.*